MATHEMATICAL ASSOCIATION OF AMERICA

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W. D. CAIRNS, *Secretary-Treasurer*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Georgetown University, Washington, D. C., on Saturday, December 6, 1941, with a morning session, luncheon, and an afternoon session. Professor E. J. McShane, chairman of the Section, presided at the sessions.

The attendance was seventy-two including the following forty-nine members of the Association: O. S. Adams, M. W. Aylor, N. H. Ball, J. D. Bankier, H. J. Barten, C. V. Bertsch, Archie Blake, C. C. Bramble, Lillian O. Brown, F. L. Celauro, C. R. Clark, Abraham Cohen, H. E. Crull, Alexander Dillingham, J. A. Duerksen, P. J. Federico, E. J. Finan, W. C. Flaherty, Michael Goldberg, T. N. E. Greville, G. A. Hedlund, L. C. Hutchinson, L. M. Kells, Evelyn M. Kennedy, Solomon Kullback, W. D. Lambert, O. E. Lancaster, A. E. Landry, Florence P. Lewis, E. J. McShane, Sister Thomas Marie Maloney, W. K. Morrill, G. W. Patterson, E. C. Phillips, O. J. Ramler, C. H. Rawlins, J. N. Rice, Irwin Roman, R. E. Root, J. B. Scarborough, Arthur Schach, E. D. Schell, W. F. Shenton, F. W. Sohon, Mary C. Varnhorn, G. C. Vedova, C. H. Wheeler III, G. T. Whyburn, R. H. Wilson.

At the invitation of the Section, Dr. L. P. Harrison, associate meteorologist of the Weather Bureau, U. S. Department of Commerce, gave an address on certain phases of meteorology.

A motion was passed expressing the appreciation of the Section to the authorities of Georgetown University for their generous hospitality.

After an address of welcome by Rev. Arthur A. O'Leary, president of Georgetown University, the following papers were read:

1. "On minimizing certain functions of triangular numbers" by Rev. E. C. Phillips, Georgetown University.

2. "Notes on the numerical evaluation of elliptic integrals" by W. D. Lambert, U. S. Coast and Geodetic Survey.

3. "Conditioned maxima and minima of functions" by Professor E. J. McShane, University of Virginia.

4. "Geometric stereograms and how to make them" by Professor W. F. Shenton, American University.

5. "On the foundations of the calculus of Eudoxus and Archimedes" by Professor G. C. Vedova, University of Maryland.

6. "Recent investigations concerning the nature of thunderstorms and their electrical manifestations" by Dr. L. P. Harrison, U. S. Weather Bureau.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Dr. Phillips discussed the functions arising from the solution of the following mathematical recreation: Given a triangle composed of $n(n+1)/2$ coins to invert it, so that its apex which was directed upward now points downward,

by moving the least possible number of coins. He endeavored to find a quadratic expression for the number, N, of coins which have to be moved in order that the new base will lie along an arbitrarily chosen line x spaces from the apex in the form $N=ax^2+bx+c$. It was found that the general value of N cannot be expressed by a single quadratic function of x but requires two distinct quadratic functions. The minimum integral values of these were determined.

2. Mr. Lambert gave two similar but not identical formulas, derived by quite different procedures, for the numerical computation of elliptic integrals of the first and second kinds for the extreme case where the modulus, k, is very nearly unity and the amplitude ϕ nearly 90°, so nearly that $k' \tan \phi > 1$. These formulas apply when standard procedures are inapplicable or inconvenient. He also discussed the computation of elliptic integrals of the third kind and found as a result of his experience with many special methods that the formulas involving Jacobi's theta functions and Jacobi's quantity q are in general most satisfactory. He also discussed tables and collections of formulas from the point of view of the computer.

4. Geometric stereograms or anaglyphes, as Vuibert called them in 1910, are geometric figures prepared in two complementary colors and viewed through glasses of the same colors. They will produce sharp black and white figures in three dimensions. Professor Shenton explained a simple method whereby an elementary school student can make successful drawings in three dimensions. He displayed drawings made by various processes.

5. Professor Vedova discussed how the attitude of the Greeks toward the infinite had influenced their conception of number and how this in turn forced the development of their integral calculus (method of exhaustions).

6. Dr. Harrison gave the results of recent investigations regarding the nature of thunderstorms. Slides were presented showing the formation of thunderheads and their electrical manifestations.

C. H. WHEELER III, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-second regular meeting of the Southern California Section of the Mathematical Association of America was held at Occidental College, Los Angeles, California, on Saturday, March 14, 1942. Professor L. J. Adams, chairman of the Section, presided.

The attendance was sixty-five, including the following thirty-eight members of the Association: C. M. Ablow, L. J. Adams, O. W. Albert, C. K. Alexander, L. D. Ames, Harry Bateman, Clifford Bell, E. T. Bell, L. T. Black, F. A. Butter, Jr., P. H. Daus, D. C. Duncan, W. H. Glenn, Jr., H. J. Hamilton, P. G. Hoel, C. G. Jaeger, G. R. Kaelin, Ada A. McClellan, G. F. McEwen, W. E. Mason, B. C. Moore, P. M. Niersbach, W. T. Puckett, Jr., H. R. Pyle, V. V. Quilliam, E. C. Rex, J. M. Robb, G. E. F. Sherwood, D. V. Steed, A. E. Taylor, C. W. Trigg, S. E. Urner, Morgan Ward, R. L. White, W. M. Whyburn, Clyde Wolfe, Euphemia R. Worthington, M. A. Zorn.

The following officers were elected for the coming year: Chairman, Morgan Ward, California Institute of Technology; Vice-Chairman, D. C. Duncan, Los Angeles City College; Program Committee, F. A. Butter, Jr., Chairman, L. T. Black, and the Secretary. The next meeting was tentatively scheduled to be held March 13, 1943, at the University of Southern California.

The following nine papers were read. The paper by Professor M. A. Zorn was an invited hour expository lecture.

1. "A boundary-value problem with conditions at k points" by Dr. C. P. Brady, Los Angeles City College, introduced by Professor Duncan.

2. "Statistics of nebulae" by Dr. G. F. N. Mulders, University of Redlands, introduced by Professor Albert.

3. "Square inscribed in a simple, closed, convex curve" by Professor H. J. Hamilton, Pomona College.

4. "Algebra—retrospect and prospect" by Professor M. A. Zorn, University of California at Los Angeles.

5. "Airplane lofting" by Professor L. J. Adams, Santa Monica Junior College.

6. "Analytic functions and conformal mapping on surfaces" by Professor H. R. Pyle, Whittier College.

7. "On certain polynomial differential operators" by E. C. Rex, University of Southern California.

8. "The part played by mathematical tables in the development of applied mathematics" by Professor Harry Bateman, California Institute of Technology.

9. "A property of Bernoulli numbers" by Carl Savit, California Institute of Technology, introduced by Professor Ward.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Dr. Brady used the method of successive approximations to establish the existence and uniqueness of a solution of a system of ordinary differential equations, with associated conditions at k points.

2. Dr. Mulders discussed the statistical treatment of nebular counts by comparing the dispersion of the nebulae with that of a random distribution, and showed that the probability of a random distribution of the external galaxies brighter than magnitude 12.7 is 1:420,000,000.

3. Professor Hamilton showed how the existence of a square inscribed in a simple, closed, convex curve followed from the consideration of a continuously turning inscribed rhombus and a limiting process.

4. Professor Zorn illustrated the principles of modern algebra by several examples from algebra and analysis. He stressed the value of the concept "indeterminate" (as opposed to "variable"), and predicted that its introduction into other fields will produce significant results.

5. Professor Adams gave a brief explanation of airplane lofting, with some of the current geometric and analytical practices used, and an indication of some of the mathematical problems which arise.

6. If we are given two surfaces S_1 and S_2 defined by their metrics and so related that the point (x, y) of S_1 corresponds to the point $(\bar{x}, \bar{y})$ of S_2 where $\bar{x}=\phi\ (x, y)$, $\bar{y}=\psi(x, y)$, we can find the functions Z and W, such that $Z=c_1$ is a minimal curve on S_1 and $W=c_2$ is a minimal curve on S_2. Professor Pyle showed that the conditions that W be an analytic function of Z are the same as those for the conformal mapping of S_2 on S_1. The conditions were found in terms of the differential parameter of the first order of differential geometry. A generalization of the Laplace equation was found in terms of the differential parameter of the second order.

7. Mr. Rex factorized the operator $f(D)=D^3+a_1D^2+a_2D+a_3$ for certain a's (which are rational integral polynomials in x). Then a particular solution was obtained for $[f(D)]y=0$.

8. As indicated by Professor Bateman, great progress was made when men interested in the stars thought about the navigation of the primitive sailing ship which was probably driven before a wind making the "right angle" with the sail. Records began to be kept of the direction of the wind and a voyage along the "right line" joining two places was probably found to be much more desirable than one along two legs of a triangle when oars had to be used part of the time. Eratosthenes of Cyrene recommended the measurement of the zenith distance of a star as an aid to the location of position. The idea was developed by Hipparchus, trigonometry was invented, and a table of chords or sines published in the Almagest of Ptolemy became well known, particularly when this book was prescribed as part of a pilot's education in the fifteenth century. Until the error in the compass was established there must have been doubts about the accuracy of the tables and, indeed, an error was located by Johannes Müller (Regiomontanus) who constructed his own tables of sines and tangents. Ten place tables with first differences were soon afterwards constructed by Georg Joachim (Rheticus) and these became of additional interest when it was claimed that a table of sines was also useful in predicting the range of a cannon ball. Tartaglia wisely dedicated his book of questions and inventions to King Henry the Eighth of England who, perhaps envious of the fame of Prince Henry the Navigator, preferred to be remembered as an authority on ballistics rather than matrimony. It soon became the fashion for a monarch to provide funds for work on tables and great mathematical progress was made when Kepler worked at the construction of new tables for Rudolph of Bohemia, when Flamsteed was appointed by Charles II of England to provide new tables for the use of English seamen and when Euler was appointed by Frederick the Great. A. Inglis in the Mathematical Gazette of 1936 has indeed surmised that Napier had the needs of navigation in mind when he invented logarithms; he may at any rate have been familiar with the work of Wright on Mercator's Projection.

9. Mr. Savit gave a proof by an elementary construction that the denominators of Bernoulli numbers recur infinitely many times.

P. H. DAUS, *Secretary*

WHAT IS ANALYSIS IN THE LARGE?

MARSTON MORSE, Institute for Advanced Study

1. Introduction. All mathematics is more or less "in the large" or "in the small." It is highly improbable that any definition of these terms could be given that would be satisfactory to all mathematicians. Nor does it seem necessary or even desirable that hard and fast definitions be given. The German terms "im Grossen" and "im Kleinen" have been used for some time with varying meanings. It will perhaps be interesting and useful to the reader to approach the subject historically by way of examples.

No proofs are given. In attempting to give the reader a conception of analysis in the large two ways are open. The first is to attempt an elementary exposition of the fundamental techniques. Unfortunately, this method of exposition is attempted much too often. The explanations given are fragmentary and give an exaggerated notion of the importance of some special technique, and no adequate notion of the subject as a whole. In a new and comprehensive field possibly the only way to give the beginner a stimulating and adequate notion of what the subject is about is to give examples and results which are themselves relatively complete. The cooperative reader can readily imagine the variety of techniques that might be used to obtain the stated results, and may himself invent new techniques, but in the presence of significant results he is less apt to be concerned with trivialities and subjective bypaths.

2. An example from differential geometry. Most of classical differential geometry is "in the small," that is, most theorems are proved merely in the a point. It is proved, for example, that in the neighborhood of a point P of a surface Σ, Σ can be referred to isothermic parameters so that neighboring P,

$$ds^2 = \lambda(u, v)\left[du^2 + dv^2\right] \tag{1.1}$$

with $\lambda(u, v) \neq 0$.* The question in the large as to what sort of closed surfaces can be represented as a whole with parameters (u, v) and ds^2 of the form (1.1) has been asked and answered in general only in recent years. It is required that there be just one curve $u = \text{const.}$ and just one curve $v = \text{const.}$ through each point. Among two-sided or orientable surfaces which admit such parameters, those of the topological type† of the torus are the only possibilities.

One could continue by asking a more general question. What sort of closed surfaces S admit a representation in terms of parameters (u, v) in such a manner that there is one and only one curve $u = \text{const.}$ and one and only one curve $v = \text{const.}$ through each point? Such a representation of S would in particular imply the existence at each point P of S of a vector tangent to the curve $u = \text{const.}$ through P. There would thus exist a field of vectors, one for each point

* Appropriate hypotheses as to the regularity of the representation of the surface must be made.

† A surface is of the topological type of the torus if it is the (1, 1) continuous image of the torus.

of P, tangent to S and P and varying continuously with P. For such a field to exist S must be the topological type of the torus.

Thus in questions as to the existence of parameter nets without singularities the controlling factors are those of topology. One can see why analysis or geometry in the large depends so heavily on topology.

3. An example from the theory of functions of a complex variable. The theorem that a function $f(z)$ of a complex variable z which has no singularities in the extended plane other than poles is a rational function of z, is a theorem in the large whose proof illustrates some of the salient characteristics of analysis in the large. One begins by representing $f(z)$ neighboring $z=z_0$ as the sum of the "principal part" of $f(z)$ at z_0 and a function analytic at z_0. This is the preliminary analysis in the small.

Upon subtracting the principal parts of $f(z)$ at each pole from $f(z)$ one obtains a function $\phi(z)$ bounded in absolute value and with at most removable singularities. According to Liouville, $\phi(z)$ is a constant. The theorem follows.

The analysis in the large comes in the proper definition of the extended plane and the proof of the Liouville theorem. Details will not be given but it will be of interest to state that the theorem of Liouville can be reduced to a theorem of topological character on the nature of vector fields.

4. Differential equations in the large. An example from the works of Henri Poincaré. It is no mere coincidence that Poincaré was the first to comprehend fully the possibilities of analysis in the large, and at the same time was the father of modern topology. Poincaré was not satisfied with the classical theory of differential equations. He wished to know something concerning the system of trajectories as a whole. He was greatly interested in the movements of the planets but found insufficient generality and completeness in the classical theory. His interest in Celestial Mechanics is in the background of all of his papers on differential equations.

Poincaré's first papers on differential equations are not pretentious in their generality, but in method they are most novel. Poincaré is concerned with an ordinary first order differential equation* defined at each point of a 2-sphere. In terms of any system of local coordinates (u, v) representing the neighborhood of a point (u_0, v_0) on the sphere the differential conditions have the form

$$\frac{du}{U(u, v)} = \frac{dv}{V(u, v)}.$$

The functions U and V are supposed real and analytic in (u, v) neighboring (u_0, v_0). Points (u_0, v_0) at which both U and V vanish are termed "singular points." These points are supposed finite in number on the sphere.

Poincaré makes certain assumptions concerning the singular points (u_0, v_0). To state these conditions we shall take (u_0, v_0) as the origin. Then U and V have

* Poincaré. Sur les courbes définies par les équations différentielles, Jour. de Liouville, 1881, 1882.

developments of the form

$$U = au + bv + \cdots,$$
$$V = cu + dv + \cdots$$

neighboring the origin. Poincaré assumes in most of his work that the roots λ_1 and λ_2 of the equation

$$\begin{vmatrix} a - \lambda, & b \\ c & d - \lambda \end{vmatrix} = 0$$

are distinct, different from 0, never pure imaginary and that neither $\lambda_1|\lambda_2$ nor $\lambda_2|\lambda_1$ is a positive integer. These conditions will be satisfied by most analytic examples.

Curves on the sphere which satisfy the differential equation are termed *characteristics*. In general characteristics are without singularity except at most when they pass through a singular point of the differential equation. Typical of analysis in the large, Poincaré's work permits a subdivision into three parts as follows:

(a). *A study of characteristics neighboring a singular point.*

(b). *The assignment of an index* ± 1 *to each singular point and the establishment of a relation between these indices. This part of the analysis would now be regarded as an essay in combinatorial topology.*

(c). *A description of the characteristics in the large with particular reference to recurrence and limiting trajectories*; *Results* (a) *and* (b) *are preliminary to* (c).

PART (a). In his study of characteristics neighboring a singular point, Poincaré shows that there are three principal kinds of singular points as follows:

"*Noeud.*" Neighboring a noeud (u_0, v_0) each characteristic tends to (u_0, v_0) with a definite limiting direction. For example the differential equation

$$\frac{du}{u} = \frac{dv}{2v}$$

has a noeud at the origin. In this example the characteristics have the form $kv = hu^2$ where h and k are constants.

"*Foyer.*" The characteristics approach such a singular point in the form of spirals, with the arc length becoming infinite. For example, the differential equation

$$\frac{du}{u - v} = \frac{dv}{u + v}$$

has a foyer at the origin with logarithmic spirals as characteristics.

"*Col.*" There are just two characteristics which tend to a col as a limiting point. For example, the equation

$$\frac{du}{u} = \frac{dv}{-v}$$

has a col at the origin. The characteristics $uv = \text{const.}$ include the two characteristics $u = 0$ and $v = 0$ passing through the origin.

PART (b). In the development (b), Poincaré assigns an index 1 to each noeud and to each foyer, and an index -1 to each col. Poincaré shows that *the sum of the indices of the of the singular points on the sphere equals* 2. Thus there must exist at least two singular points.

A closed characteristic without a multiple point is called a *cycle*. If a characteristic tends to a noeud or a foyer as a limit point there is in general no natural way to continue the characteristic, and it is agreed that in such cases the characteristic shall end at the noeud or foyer. If a characteristic g tends to a col the convention is made that g may be continued turning either to the right or left and departing from the col on a characteristic. By virtue of this convention the notion of a cycle is enlarged. With this understood we see that a cycle can have no singularity other than those occurring at a col.

PART (c). Poincaré ends with a relatively complete description of the characteristics. He shows that *a characteristic continued without limit in a given sense either terminates at a noeud, or is a cycle, or is asymptotic to a cycle.* A foyer is to be regarded as a degenerate cycle to which the neighboring spirals are asymptotic.

The reader is asked to observe the fundamental difference between the modes of analysis required in Parts (a), (b), and (c), and then to note how (a) and (b) are preliminary to (c) and make (c) possible. The index theorem of Poincaré has its topological generalization in the fixed point theorems of Brouwer, Alexander, Lefschetz, and H. Hopf. The analysis of characteristics in (c) is the predecessor of the modern study of recurrence and transitivity which G. D. Birkhoff has developed so fully and to which Hedlund, Morse, von Neumann, Koopman, E. Hopf and others have made significant contributions.

5. Elementary examples in equilibrium theory in the large. Equilibrium theory in the large makes an extensive use of topology. The principles of analysis brought out in the previous examples appear here again. Briefly summed we have seen in these examples that analysis in the large has involved (a) a preliminary analysis in the small, (b) a local determination of indices, (c) an integration of this local analysis by various means (including topology) into the final theorems in the large. The examples which we shall now present will show how various problems which from a local point of view appear most diverse, from a topological point of view are essentially the same.

We begin with certain results concerning a function f of a point on a closed bounded n-manifold Σ lying in an euclidean space of sufficiently high dimension. We suppose throughout that Σ is locally represented in terms of n parameters (u) with convenient conditions of differentiability and regularity. In terms of the local parameters (u) f shall be a function $F(u)$ at least three times continu-

ously differentiable. A *critical* or *equilibrium point* of f is a point at which each partial derivative of F is null.

For the purposes of this exposition we shall make an assumption which is in general fulfilled. We shall suppose that each critical point is *non-degenerate* in the sense that the terms F_2 of the second order in the Taylor's formula for F about the critical point is a non-degenerate quadratic form. Then, as in the elementary theory of conic sections it is possible to make a real non-singular linear transformation from the variables (u) to the variables (v) such that F_2 takes the form

$$F_2 = -v_1^2 - \cdots - v_k^2 + v_{k+1}^2 + \cdots + v_n^2$$

The number k is called the *index* of the critical point.

A manifold such as Σ possesses an ith Betti number R_i $(i=1, \cdots, n)$. This is the maximum number of independent non-bounding i-cycles* on Σ. For example, if Σ is a torus then $R_0=1$, $R_1=2$, $R_2=1$. We shall be concerned with a 3-dimensional torus T_3. Such a manifold can be obtained by starting with a 2-dimensional torus T_2 and a 2-plane π_2 lying in a euclidean 3-plane, with π_2 not intersecting T_2. To obtain T_3 we revolve T_2 about π_2 in a 4-plane containing our 3-plane. Such a T_3 is sometimes called a product of three circles. For T_3 one has $R_0=1$, $R_1=3$, $R_2=3$, $R_3=1$. These numbers are the binomial coefficients when $n=3$. We can obtain a 1–1 continuous image of an ordinary torus by identifying opposite sides of a square. Similarly one can obtain a 1–1 continuous image of T_3 by identifying opposite faces of a cube. With this identification three mutually perpendicular edges of the cube represent three independent non-bounding 1-cycles, as can be shown. Similarly three mutually perpendicular faces of the cube represent three independent non-bounding 2-cycles. A point is a 0-cycle and T_3 itself is a 3-cycle. In this way one intuitively accounts for the fact that $R_0=1$, $R_1=3$, $R_2=3$, $R_3=1$. A 3-dimensional manifold which is a 1–1 continuous image of T_3 will be called a *topological* 3-torus.

The theorem which will be used in what follows is that on Σ the number M_i of critical points of f of index i satisfies the fundamental relation†

$$M_i \geqq R_i. \tag{5.1}$$

Thus on a topological 3-torus one can infer the existence of at least $1+3+3+1=8$ critical points.

Example 5.1. *Triangles of light.* Let there be given three non-intersecting, simple, closed, non-singular, analytic curves C_1, C_2, C_3 all lying in a 2-plane. We shall be concerned with triangles with vertices p_1, p_2, p_3 on C_1, C_2, C_3 respectively. Such a triangle will be called a *triangle of light* if a ray of light following

* For details see Seifert-Threlfall, Lehrbuch der Topologie, Leipzig, 1934. Chap. III.

† See Morse, Calculus of variations in the large. Colloquium Lectures. Amer. Math. Soc. (1934) Chap. VI.

Also, Seifert-Threlfall. Variationsrechnung im Grossen, Leipzig, 1938.

this triangle is reflected at p_i as if C_i were a mirror, or if the angle in the triangle at p_i is π. How many triangles of light can we affirm to exist?

Let f be the sum of the lengths of the sides of the triangle $p_1p_2p_3$. We can refer C_i to a parameter u_i which is proportional to the arc length and varies from 0 to 2π. Then f becomes a function $f(u_1, u_2, u_3)$. The domain of definition of f is clearly a topological 3-torus. As a matter of analysis in the small one proves by elementary methods that f has a critical point if and only if the corresponding triangle is a triangle of light.

These triangles of light can then be classified according to the index of the corresponding critical point. According to relation (5.1) in the general theory of critical points there are at least $8=1+3+3+1$ of these triangles of light.

Example 5.2. *Normals from a point to a topological 3-torus.* Let Σ_3 be a topological 3-torus in a euclidean 4-space. Let p be a fixed point not on Σ_3. We seek normals from p to Σ_3. To obtain these we let f be the distance from p to Σ_3 regarding f as a function of the point (u) of Σ_3. It can be shown that except for a subset of special points p the critical points of f are non-degenerate. Moreover one then shows by a local analysis that f has a critical point (u) if and only if the line segment from p to (u) is normal to Σ_3 at (u). According to our general theorem there are then at least eight normals from p to Σ_3. These normals can be classified and it can be shown that the index of a non-degenerate critical point is the number of centers of principal curvature of Σ_3 between p and (u) on the given normal. Similar theorems hold for a topological 2-torus. Here the number of normals is at least 4.

Example 5.3. *3-planes passing through a fixed 2-plane and tangent to the preceding topological 3-torus* Σ_3. We suppose that the fixed 2-plane π_2 does not pass through a hole in Σ_3, that is, we suppose that π_2 can be moved indefinitely away from Σ_3 without intersecting Σ_3. We can then show that if π_2 is non-specialized there are at least eight 3-planes through π_2 tangent to Σ_3.

Example 5.4. *Heavy chain in equilibrium, with ends free to move on a topological 2-torus and on a closed curve C respectively.* We suppose that the curve C and the topological 2-torus Σ_2 lie in euclidean three space but that no point of C and Σ_2 lie on the same plumb line. We suppose that a chain is provided which is larger than the maximum distance from a point of C to a point of Σ_2. The end points of the chain are supposed free to move on Σ_2 and C respectively and the chain is permitted to pass through Σ_2 or C. If the position of C is non-specialized relative to Σ_2, then there are at least eight positions of equilibrium of the chain, seven of which are unstable. The function whose critical points are sought gives the height of the center of gravity of the chain as a function of the end points of the chain.

The examples of this section are unified by the fact that the function involved is defined in each case on a topological 3-torus. The examples belong equally well to mechanics, geometry, or the calculus of variations. The restrictions as to non-specialized positions of the configurations involved can all be removed by replacing the definition of a critical point in terms of derivatives by a topological

definition of a critical point, and by replacing the classification of critical points according to their indices by a topological classification of a group theoretic character. This type of generalization both in its form and genesis is characteristic of analysis in the large.

It is possible that analysis in the large may eventually reduce to topology, but not until topology has been greatly broadened. It is equally conceivable that the apparently less general situations which arise with such frequency in problems in analysis in the large may form the canonical cases about which the topology of the future can be built.

Analysis is full of difficult but significant unsolved problems in the large. We mention only one example. How does the topological structure of the contour manifolds of the Jacobi least action integral J in the problem of three or more celestial bodies vary with the value of J? The independent variable in J is a closed path. The solution of this problem may disclose that the planetary orbits exist for essentially topological reasons. On the purely topological side the number of problems whose solution is necessary for a rapid advance of analysis in the large is very great, presenting a field that is virtually untouched.

ON PORISTIC QUADRILATERALS

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1. Introduction. In 1828 Jacobi [1] found the condition that a closed polygon of n sides can be inscribed in one circle and circumscribed about another. Special cases of this problem had been solved prior to 1828 by Fuss [2] and Poncelet [3]. Later Cayley [4] generalized the problem by considering polygons inscribed in one ellipse and circumscribed about another. W. E. Byerly [5] gave some interesting constructions and his paper has further references.

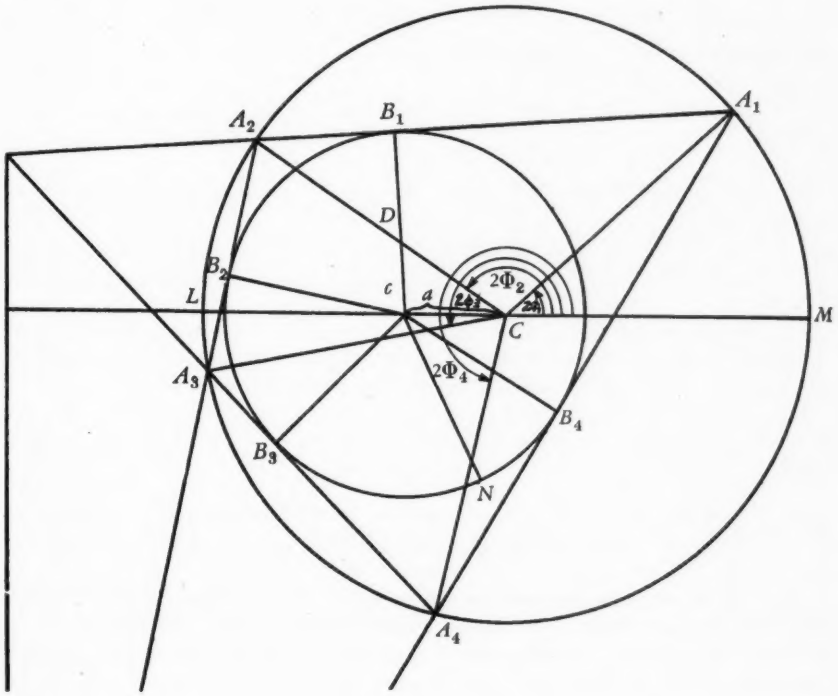
Jacobi used the theory of elliptic functions to obtain his general solution of the problem. Since the elementary theory of elliptic functions is well within the range of information of students of advanced calculus, it seems that a review of Jacobi's results with a few new theorems may be of interest.

2. Definition and relations. A poristic polygon is one which is inscribed in one circle (ellipse) and circumscribed about another. In the adjoining figure $A_1A_2A_3A_4$ is such a polygon. We denote the radii of the two circles by R and r and the distance between their centers by a. Let us choose an arbitrary point, A_1, on the larger circle as a starting point and draw the line A_1A_2 tangent to the smaller circle. Let the centers of the two circles be C and c. The diameter LM passes through c and C. Let CA_1 make an angle $2\phi_1$ with LM. The lines CA_2, CA_3, CA_4 make the angles $2\phi_2$, $2\phi_3$, and $2\phi_4$ respectively with LM. We shall use C as the origin and LM as the x-axis of a rectangular system of coördinates whenever we need to write the equations of any line or the coordinates of a point.

From the figure and from the above definitions, the following simple relations are easily obtained:

The acute angle at D is $\phi_2-\phi_1$, the angle B_1cC is $\phi_2+\phi_1$,

$$cB_1 = R\cos(\phi_2 - \phi_1) = a\cos(\phi_2 + \phi_1).$$



The last relation can be written in the form

$$\frac{r}{R+a} = \cos\phi_2\cos\phi_1 + \frac{R-a}{R+a}\sin\phi_2\sin\phi_1.$$

Let

$$\cos\alpha = \frac{r}{R+a}, \text{ then } 1 - k^2\sin^2\alpha = \frac{(R-a)^2}{(R+a)^2} \text{ where } k^2 = \frac{4aR}{(R+a)^2 - r^2}.$$

One can easily prove that $k^2<1$ if $r<R-a$. Then we have

$$\cos\alpha = \cos\phi_2\cos\phi_1 + \sqrt{1 - k^2\sin^2\alpha}\sin\phi_2\sin\phi_1. \tag{1}$$

3. Lagrange's formula. At this point Jacobi must have noticed the similarity between equation (1) and Lagrange's formula. In fact, Lagrange's formula is

$$\operatorname{cn}(u - v) = \operatorname{cn} u \operatorname{cn} v + \operatorname{sn} u \operatorname{sn} v \operatorname{dn}(u - v),$$

and if we substitute $v = u + T$, we have

$$\operatorname{cn} T = \operatorname{cn} u \operatorname{cn}(u + T) + \operatorname{sn} u \operatorname{sn}(u + T)\operatorname{dn} T. \tag{2}$$

Equations (1) and (2) are identical if

$$\alpha = \operatorname{am} T \quad \text{or} \quad T = \int_0^{\alpha} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

$$\phi_1 = \operatorname{am} u \quad \text{or} \quad u = \int_0^{\phi_1} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

$$\phi_2 = \operatorname{am}(u + T) \text{ or } u + T = \int_0^{\phi_2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

4. The closed polygon. When the second tangent is drawn with one end at A_2 and the other at A_3 it follows that ϕ_3 is related to ϕ_2 just as ϕ_2 is related to ϕ_1. Thus $\phi_3 = \operatorname{am}(u+2T)$. And in general $\phi_{n+1} = \operatorname{am}(u+nT)$. Now if the $n+1$ vertex is made to coincide with A_1 we shall have a closed polygon and in order that this be the case we must have $\phi_{n+1} = \phi_1 + h\pi$ where h is a positive integer. Hence,

$$u + nT = \int_0^{\phi_1 + h\pi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^{\phi_1} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} + \int_0^{h\pi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

From well known theorems on elliptic integrals it follows that $u + nT = u + 2hK$ and $nT = 2hK$ where K is the complete elliptic integral of the first kind. This is Jacobi's criterion for a closed polygon of n sides. If the polynomial is a quadrilateral $n = 4$ and $h = 1$ so that

$$T = \frac{K}{2}. \tag{3}$$

Jacobi remarked that, since this condition is independent of ϕ_1, it follows that that starting point is quite arbitrary, and we may get as many closed poristic polygons as we please by varying the starting point.

The condition (3) may be expressed in terms of an elliptic integral in the form

$$\frac{K}{2} = \int_0^{\alpha} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

5. A relation between R, r, and a. Equation (3) together with $r < R < a$ are necessary conditions for a closed poristic quadrilateral. Further, equation

(2) is an identity in u. Hence, if we can draw one quadrilateral, we can draw as many as we please. For convenience we choose $u=K/2$. Noting that sn $K=1$ and cn $K=0$ equation (2) reduces to

$$\text{cn } K/2 = \text{sn } K/2 \text{ dn } K/2.$$

Since

$$\text{cn } K/2 = \text{cn } T = \cos\alpha = \frac{r}{R+a}$$

we have

$$\frac{r}{R+a} = \sqrt{1 - \frac{r^2}{(R+a)^2}}\,\frac{R-a}{R+a}.$$

which easily reduces to

$$\frac{r^2}{(R+a)^2} + \frac{r^2}{(R-a)^2} = 1. \tag{4}$$

It follows that

$$\sin\alpha = \frac{r}{R-a}.$$

6. Sufficient conditions. Equation (3) and $r<R-a$ are necessary for a closed poristic quadrilateral. We shall also prove that they are sufficient. For suppose that they are satisfied. Then (4) is true. Draw two tangent lines, below the x-axis, to the smaller circle, one passing through M and the other through L. Draw the radii to the point of contact. Then the angle McN is the angle α. The acute angle at L is also equal to α since $\sin\alpha=r/(R-a)$. It follows that the tangent line passing through L is perpendicular to the tangent line through M. Hence they must meet on the circumference of the larger circle. Therefore, if (4) is satisfied we can draw at least one poristic quadrilateral. Hence if (4) and (3) are satisfied we can draw as many as we please.

If the poristic polygon is a triangle the relation corresponding to (4) is

$$\frac{r}{R+a} + \frac{r}{R-a} = 1.$$

These two relations and corresponding equations for poristic pentagons and hexagons were known before Jacobi's paper of 1828, but the general conditions for a closed polygon were first given by Jacobi.

7. The quadrilateral $B_1B_2B_3B_4$. Let B_1, B_2, B_3, B_4 be the points at which the sides of the polygon $A_1A_2A_3A_4$ are tangent to the smaller circle. We shall now prove a theorem about the diagonals of the quadrilateral $B_1B_2B_3B_4$. The coordinates of B_1, B_2, B_3 and B_4 are,

$$[r\cos(\phi_1+\phi_2)-a,\ r\sin(\phi_1+\phi_2)],$$
$$[r\cos(\phi_2+\phi_3)-a,\ r\sin(\phi_2+\phi_3)],$$
$$[r\cos(\phi_3+\phi_4)-a,\ r\sin(\phi_3-\phi_4)],$$
$$[-r\cos(\phi_4-\phi_1)-a,\ -r\sin(\phi_4+\phi_1)].$$

The slopes of B_1B_3 and of B_2B_4 are easily found to be $-\cot S/2$ and $\tan S/2$ where $S=\phi_1+\phi_2+\phi_3+\phi_4$. Hence the equations of the internal diagonals of $B_1B_2B_3B_4$ are

$$y-r\sin(\phi_1+\phi_2)=-[x+a-r\cos(\phi_1+\phi_2)]\cot S/2,$$
$$y-r\sin(\phi_2+\phi_3)=[x+a-r\cos(\phi_2+\phi_3)]\tan S/2$$

We are interested in the point of intersection of B_1B_3 and B_2B_4. Solving the equations we find the coördinates to be given by

$$2(x+a)\csc S=r\sin(\phi_1+\phi_2)-r\sin(\phi_2+\phi_3)+r\cos(\phi_1+\phi_2)\cot S/2$$
$$+r\cos(\phi_2+\phi_3)\tan S/2,$$

and

$$2y\csc S=r[\cos(\phi_1+\phi_2)-\cos(\phi_2+\phi_3)+\sin(\phi_2+\phi_3)\cot S/2$$
$$+\sin(\phi_1+\phi_2)\tan S/2].$$

In order to simplify these expressions we make use of some fundamental relations given by Jacobi on page 285 of his collected works. The relations we need are

$$\tan\frac{\phi_1+\phi_3}{2}=\sqrt{k_1}\tan\phi_2,\qquad \tan\frac{\phi_2+\phi_4}{2}=\sqrt{k_1}\tan\phi_3,$$
$$\tan\frac{\phi_3+\phi_5}{2}=\sqrt{k_1}\tan\phi_4,\qquad k_1^2=1-k^2=\left(\frac{R-a}{R+a}\right)^2.$$

In the closed quadrilateral $\phi_5=\pi+\phi_1$. Hence the third relation above becomes, by using the first,

$$-\cot\phi_2=k_1\tan\phi_4.$$

The above formulas enable us to write $\tan S/2$ in the form

$$\tan S/2=\tan\left\{\frac{\phi_1+\phi_3}{2}+\frac{\phi_2+\phi_4}{2}\right\}=\frac{\sqrt{k_1}\sin(\phi_2+\phi_3)}{\cos\phi_2\cos\phi_3-k_1\sin\phi_2\sin\phi_3}.$$

But from the addition formulas for the elliptic functions we have

$$\sin\phi_3=\operatorname{sn}(u+K)=\frac{\operatorname{cn}u}{\operatorname{dn}u}=\frac{\cos\phi_1}{\operatorname{dn}u},$$

and

$$\cos\phi_3=-\frac{k_1\sin\phi_1}{\operatorname{dn}u}.$$

Hence

$$-\cot \phi_1 = k_1 \tan \phi_3.$$

This enables us to write

$$\tan S/2 = \frac{k_1 \sin \phi_1 \sin \phi_2 - \cos \phi_1 \cos \phi_2}{\sqrt{k_1} \sin (\phi_1 + \phi_2)}.$$

In the equation which gives y above we replace cot $S/2$ by the reciprocal of the first form for tan $S/2$ and replace tan $S/2$ by its second form, then immediately all the terms in the right member cancel and therefore $y=0$.

To find a simpler form for x we multiply by sin S, combine the proper terms, and remove the cosines by the formula

$$\cos \phi_i \cos \phi_{i+1} = -\sqrt{k_1} \sin \phi_i \sin \phi_{i+1} + \frac{r}{R+a}, \qquad i = 1, 2, 3.$$

There results

$$\left\{\frac{2(x+a)}{r} - \frac{4r}{R+a}\right\} \frac{1}{1+\sqrt{k_1}} = -\sin \phi_2(\sin \phi_1 + \sin \phi_3) + \sin \phi_4(\sin \phi_1 - \sin \phi_3),$$

or

$$\frac{2(x+a)(R+a) - 4r^2}{2Rr} = -\operatorname{sn}(u+T)\left(\operatorname{sn} u + \frac{\operatorname{cn} u}{\operatorname{dn} u}\right) + \frac{\operatorname{cn}(u+T)}{\operatorname{dn}(u+T)}\left(\operatorname{sn} u - \frac{\operatorname{cn} u}{\operatorname{dn} u}\right).$$

If we expand sn $(u+T)$, cn $(u+T)$, and dn $(u+T)$ and simplify, all of the terms containing elliptic functions of u cancel and the right member reduces to -2 sn $K/2$. Hence

$$x = -\frac{2ar^2}{R^2 - a^2} - a.$$

THEOREM. *The internal diagonals of the quadrilateral $B_1B_2B_3B_4$ are mutually perpendicular and intersect in a fixed point on the line of centers of the two circles.*

8. The external diagonal of $A_1A_2A_3A_4$. The external diagonal of the quadrilateral $A_1A_2A_3A_4$ is the polar of the point of intersection of the internal diagonals of the quadrilateral $B_1B_2B_3B_4$. In the notation which we have employed the equation of the external diagonal is

$$x = -\frac{R^2 + a^2}{2a}.$$

Since this is independent of ϕ_1 it follows that the points of intersection of the opposite sides of the poristic quadrilateral $A_1A_2A_3A_4$ move along the line as ϕ_1 varies.

THEOREM. *The external diagonal of all the poristic quadrilaterals that can be drawn to two fixed circles is a fixed straight line perpendicular to the line of centers of the two circles.*

This theorem enables us to draw as many different poristic quadrilaterals as we please with very little labor. We first choose R and a then compute the right member of the equation of the external diagonal and draw the line. After the two circles have been drawn we choose any point on the external diagonal and draw two tangent lines to the smaller circle. These tangent lines cut the larger circle in the vertices of a poristic quadrilateral.

9. Symmetric quadrilaterals. If we start from the point M and draw a poristic quadrilateral then $\phi_1=0$ and am $u=0$. Since $\phi_3=\text{am}\ (u+2T)$ it follows that $\phi_3=\pi/2$, that is, the end point of the second tangent line is at the end of the diameter opposite to M. Since we can measure ϕ_2 and ϕ_3 clockwise as well as counterclockwise it follows that every poristic quadrilateral which has one vertex at M is symmetrical with respect to the line of centers of the two circles.

10. Poristic quadrilaterals for two ellipses. Let us draw a plane making the angle θ with the plane of the two circles of §2. This plane may pass through the external diagonal of the quadrilateral $A_1A_2A_3A_4$. Project the figure on this plane by drawing perpendiculars from all of its points to the new plane. Then it is obvious that the two circles project into ellipses; that the projections of the diagonals of $B_1B_2B_3B_4$ will still intersect on the line of centers of the two ellipses and that the theorem of §8 remains true in the projected figure.

11. The quadrilateral $B_1B_2B_3B_4$. We have already proved that the internal diagonals of the quadrilateral $B_1B_2B_3B_4$ intersect on the x-axis and at a fixed point. They are also mutually perpendicular. Now we rotate the quadrilateral $A_1A_2A_3A_4$, by varying ϕ_1, so that the point B_2 is on the x-axis. When this is done the diagonal B_2B_4 must coincide with the x-axis, so that B_2B_4 is a diameter of the smaller circle. The diagonal B_1B_3 is perpendicular to B_2B_4 and is bisected by it. Hence $B_1B_2B_3B_4$ is symmetrical to the x-axis and it is obvious that a third circle can be inscribed in it. Hence the quadrilateral $B_1B_2B_3B_4$ is a poristic quadrilateral. The center of the third circle may be determined by bisecting the interior angle at B_1. We may determine the coordinates of the center by computing the lengths of B_1B_2 and B_1B_4 and using the theorem that the bisector of any angle of a triangle divides the opposite side into segments which are in the same ratio as the adjacent sides. If the segments of B_2B_4 are $2r-x$ and x we find

$$\frac{2r-x}{x}=\tan\,(\phi_1+\phi_2)/2 \quad \text{or} \quad x=\frac{2r}{[1+\tan\,(\phi_1+\phi_2)/2]}.$$

The coördinates of the center of the third circle turn out to be

$$\left\{ -r\frac{[1-\sin(\phi_1+\phi_2)]}{\cos(\phi_1+\phi_2)} - a, 0 \right\}.$$

We can simplify still further the expressions $\sin(\phi_1+\phi_2)$ and $\cos(\phi_1+\phi_2)$. We notice that B_2 and B_4 are both on the x-axis and $\cos 2\phi_1 = (r-a)/R$, or $\sin\phi_1 = \sqrt{(R-r+a)/2R}$ and $\cos\phi_1 = \sqrt{(R+r-a)/2R}$, $\cos 2\phi_2 = -(r+a)/R$, or $\sin\phi_2 = \sqrt{(R+r+a)/2R}$, and $\cos\phi_2 = \sqrt{(R-r-a)/2R}$. Then, in terms of R, r, and a, the coördinates of the center of the third circle are

$$\left[\left\{ \frac{\sqrt{(R-r)^2-a^2}+\sqrt{(R+r)^2-a^2}}{4aR} \right\} (R^2-a^2) - a, 0 \right].$$

This last expression is not very simple. It is given for the purpose of showing that the coördinates of the third circle may be calculated in terms of R, r, and a.

We can now state the theorem: *Every poristic quadrilateral, $A_1A_2A_3A_4$ determines another poristic quadrilateral $B_1B_2B_3B_4$.*

It is, of course, obvious that we may continue this process indefinitely.

Let us change the notation so that the radius of the first circle is R_1, of the second R_2 and so on. Let the distance between the centers of the first and second circles be a_1, between the centers of the second and third be a_2, and so on. Then we always have the relation

$$\left(\frac{R_{i+1}}{R_i + a_i}\right)^2 + \left(\frac{R_{i+1}}{R_i - a_i}\right)^2 = 1.$$

One cannot help but wonder what happens to this relation as i increases indefinitely. It seems probable that a_i approaches 0 and that R_{i+1} approaches $R_i/\sqrt{2}$.

We have made no attempt to apply these results to the case, which Byerly discusses, in which the second circle is entirely outside the first. It would be interesting to follow the discussion through in that case.

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2. Nicolaus Fuss, Nova Acta. Petersburg, Band 13, pp. 166–189.
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REMARKS ON TERNARY DIOPHANTINE EQUATIONS

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Let $F(x, y)$ be a binary form of order $n \geqq 3$ with integral coefficients, $k \neq 0$ an integer, $P_1, \cdots, P_t$ a finite set of different prime numbers. The following results have been proved:*

THEOREM 1. *If the equation*

$$F(x, y) = k$$

has an infinity of integral solutions x, y, then $F(x, y)$ is a power of a linear form or of an indefinite quadratic form.

THEOREM 2. *If the equation*

$$F(x, y) = \mp P_1^{z_1} \cdots P_t^{z_t}$$

has an infinity of integral solutions $x, y, z_1, \cdots, z_t$, where x and y are relatively prime, and $z_1 \geqq 0, \cdots, z_t \geqq 0$, then $F(x, y)$ is a power of a linear or quadratic form.

Now consider a ternary form $F(x, y, z)$ of order $n \geqq 3$ with integral coefficients, and the representations of integers $k \neq 0$ by this form. If $F(x, y, z)$ is decomposable into a product of linear forms with algebraic coefficients and if n is sufficiently large, then results analogous to Theorem 1 and 2 are true.† On the other hand, very little is known about the more general case when $F(x, y, z)$ is irreducible in the field of all constants. In this note, I construct examples of ternary forms of every order with the property of representing at least one integer $k \neq 0$, or even every integer, in an infinity of different ways. I further show how to form positive definite ternary forms of every even order with the property that for an infinity of different sets of relatively prime integers x, y, z the greatest prime factor of $F(x, y, z)$ is bounded. Special cases of forms with these properties are well known; *e.g.*, the equation

$$x^3 + y^3 + z^3 = 1$$

has an infinity of integral solutions, since identically in t,

$$(9t^4)^3 + (3t - 9t^4)^3 + (1 - 9t^3)^3 = 1;$$

and the equation

$$x^4 + y^4 + z^4 = 2 \cdot 7^{z_1}$$

has an infinity of integral solutions with relatively prime x, y, z, since identically in x and y,‡

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* A. Thue, Norske Vid. Selsk. Skr. 1908, Nr. 7. K. Mahler, Math. Ann. 107, 1933, pp. 691–730.

† C. L. Siegel, Math. Z. 10, 1921, pp. 173–213.
E. T. Parry, Journal of the London Mathematical Society, 1940, vol. 15, pp. 293–305.

‡ The equation $x^3+xy+y^2=7^z$ has an infinity of integral solutions with relatively prime x, y.

$$x^4 + y^4 + (x + y)^4 = 2(x^2 + xy + y^2)^2.$$

The stated results are obtained by the construction of simple identities. In a similar way, it is possible to show the following theorem: "*If $F(x, y, z)$ is an irreducible cubic form with integral coefficients, such that the equation $F(x, y, z) = 0$ has at least one solution in integers not all zero, then $F(x, y, z)$ either represents all integers in a suitable linear progression $at+b$ $(t = 0, \mp 1, \mp 2, \cdots)$ or it represents a certain integer $k \neq 0$ in an infinity of different ways.*" Let g, h, k be three integers not all zero such that

$$F(g, h, k) = 0,$$

and denote with F_g, F_h, F_k the values of the three first partial derivatives $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$ for $x = g$, $y = h$, $z = k$. There are three integers not all zero such that

$$GF_g + HF_h + KF_k = 0.$$

Let now t be a parameter; then

$$F(gt + G, ht + H, kt + K) = At^3 + Bt^2 + Ct + D$$

is a cubic polynomial in t. This polynomial cannot vanish identically, since $F(x, y, z)$, by hypothesis, is irreducible. It is however at most of the first degree. For the assumptions about g, h, k; G, H, K are equivalent to $A = B = 0$. According as to whether $C \neq 0$ or $C = 0$, F represents all integers of the progression $Ct+D$, or is equal to D for all values of t. In the second case $D \neq 0$, since $F \not\equiv 0$.

1. Ternary equations with an infinity of solutions. The general ternary form $F(x, y, z)$ of order n has

$$N = \frac{(n + 1)(n + 2)}{2}$$

coefficients. If

$$p_1(t), \quad p_2(t), \quad p_3(t) \qquad \left(\sum_{h=1}^{3} | p_h(t) | > 0 \text{ for all } t \right)$$

are three polynomials in a parameter t with integral coefficients and of degree less than or equal to ν, then

$$F(p_1(t), p_2(t), p_3(t)) = \phi(t)$$

is a polynomial in t of degree not greater than $n\nu$; its coefficients are linear forms in the coefficients of $F(x, y, z)$ with integral numerical coefficients; say

$$\phi(t) = \sum_{h=0}^{n\nu} L_h(F) t^h.$$

If $\phi(t)$ is to be a constant, then the coefficients of $F(x, y, z)$ must satisfy the $n\nu$ linear equations

$$L_h(F) = 0 \qquad (h = 1, 2, \cdots, n\nu).$$

For $N > n\nu$, this system has always a non-trivial solution in integers; there is therefore then a ternary form of order n with integral coefficients not all zero such that

$$F(p_1(t), p_2(t), p_3(t)) = L_0(F)$$

is independent of t. The so constructed form $F(x, y, z)$ may, however, be reducible, and the constant $L_0(F)$ on the right-hand side may vanish. In the special case $\nu = 1$ of linear polynomials $p_h(t)$ both complications can always be avoided, and it is possible to determine irreducible forms $F(x, y, z)$ of every order n such that the constant $\phi(t) = L_0(F) \neq 0$.*

The form $F(x, y, z)$ constructed in this manner is *not* definite; it assumes the values $k = L_0(F)$ for every integral value $t = 0, \mp 1, \mp 2, \cdots$. I give here a few examples of this kind:

$x^3+y^3+z^3+\lambda xyz = \lambda^3+27$ identically in t for $x = t+\lambda$, $y = -t$, $z = 3$;

$y^2z-12x^3+3z^3 = 12$ identically in t for $x = t+1$, $y = 3t$, $z = t+2$;

$2x^4-y^4-z^4+2x^2y^2+2x^2z^2-4y^2z^2 = -6$ identically in t for $x = t$, $y = t+1$, $z = t-1$;

$2(x^4+y^4)-(x-y)^2z(3x-3y+z) = 4$ identically in t for $x = t+1$, $y = t-1$, $z = t^2$.

2. Forms which represent every integer. Let α, β, γ be three integers and assume that the form $F(x, y, z)$ of order n has the following property:

"On replacing z by $\alpha x+\beta y+\gamma$, we get identically in x and y,

$$F(x, y, \alpha x + \beta y + \gamma) = p(x) + ay, \tag{1}$$

where a is a constant and $p(x)$ a polynomial in x both depending on F."

Assume the form has this property, and let

$$\xi_1, \xi_2, \cdots, \xi_s$$

be the different residues of $p(x)$ mod a; since

$$p(x) \equiv p(x') \bmod a \quad \text{for} \quad x \equiv x', \bmod a,$$

each congruence

$$p(x) \equiv \xi_\sigma \bmod a \qquad (\sigma = 1, 2, \cdots, s)$$

has an infinity of integral solutions x. Hence if k is any integer in one of the residue classes

$$k \equiv \xi_\sigma \bmod a \qquad (\sigma = 1, 2, \cdots, s),$$

then

$$p(x) + ay = k$$

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* This corresponds to the fact that there are irreducible algebraic curves of every order n which intersect a given straight line only in n coinciding points.

has an infinity of integral solutions x, y, and therefore the equation $F(x, y, z) = k$ has an infinity of integral solutions, $x\ y,\ z$.

The following forms satisfy the identity (1):

$$(2)\qquad \begin{aligned} F_l(x, y, z) &= x^{n-l}(z - \beta y)^l \qquad (l = 0, 1, \cdots, n-1);\\ F_n(x, y, z) &= y(z - \alpha x - \beta y)^{n-1}, \end{aligned}$$

as is evident on putting $z = \alpha x + \beta y + \gamma$. A further and less obvious solution is obtained in the following way:

We assume that this solution can be written as

$$(3)\qquad \begin{aligned} F_{n+1}(x, y, z) = z^n &+ yf_1(x, y)z^{n-2} + yf_2(x, y)z^{n-3} + \cdots\\ &+ yf_{n-2}(x, y)z + yf_{n-1}(x, y), \end{aligned}$$

where $f_k(x, y)$, for $k = 1, 2, \cdots, n-1$, is a binary form in x, y of order k with integral coefficients. Evidently none of the forms (2) is of this type.

From (1) and (3), we get

$$\begin{aligned} F_{n+1}(x, y, \alpha x + \beta y + \gamma) &= (\alpha x + \beta y + \gamma)^n + yf_1(x, y)(\alpha x + \beta y + \gamma)^{n-2} + \cdots\\ &\quad + yf_{n-1}(x, y)\\ &= p(x) + ay. \end{aligned}$$

Hence, on developing both sides into sums of binary forms in x, y and comparing forms of equal order,

$$\begin{aligned} &(\alpha x+\beta y)^n+yf_1(x, y)(\alpha x+\beta y)^{n-2}+\cdots+yf_{n-2}(x, y)(\alpha x+\beta y)+yf_{n-1}(x, y) &&= c_n x^n,\\ &\binom{n}{1}(\alpha x+\beta y)^{n-1}+\binom{n-2}{1}yf_1(x, y)(\alpha x+\beta y)^{n-3}+\cdots+\binom{1}{1}yf_{n-2}(x, y) &&= c_{n-1}x^{n-1},\\ &\binom{n}{2}(\alpha x+\beta y)^{n-2}+\binom{n-2}{2}yf_1(x, y)(\alpha x+\beta y)^{n-4}+\cdots+\binom{2}{2}yf_{n-2}(x, y) &&= c_{n-2}x^{n-2},\\ &\cdots\cdots\cdots

$$\binom{n-2}{2} yf_1(x, y)(\alpha x+\beta y)^{n-4}+\binom{n-3}{2} yf_2(x, y)(\alpha x+\beta y)^{n-5}+\cdots+\binom{2}{2} yf_{n-3}(x, y)=\binom{n}{2} d_{n-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\binom{n-2}{n-2} yf_1(x, y) = \binom{n}{n-2} d_2.$$

This system of linear equations for

$$yf_1(x, y),\ yf_2(x, y),\ \cdots,\ yf_{n-1}(x, y)$$

is easily solved just in the required form; for the expressions on the right-hand side are all divisible by y. We find for small values of n:

$n = 1$: $F_2(x, y, z) = z$;

$n = 2$: $F_3(x, y, z) = z^2 + d_2$;

$n = 3$: $F_4(x, y, z) = z^3 + 3d_2z + (d_3 - 3(\alpha x + \beta y)d_2)$;

$n = 4$: $F_5(x, y, z) = z^4 + 6d_2z^2 + 4(d_3 - 3(\alpha x + \beta y)d_2)z + (d_4 - 4(\alpha x + \beta y)d_3 + 6(\alpha x + \beta y)^2d_2)$;

$n = 5$: $F_6(x, y, z) = z^5 + 10d_2z^3 + 10(d_3 - 3(\alpha x + \beta y)d_2)z^2 + 5(d_4 - 4(\alpha x + \beta y)d_3 + 6(\alpha x + \beta y)^2d_2) + (d_5 - 5(\alpha x + \beta y)d_4 + 10(\alpha x + \beta y)^2d_3 - 10(\alpha x + \beta y)^3d_2)$.

It is also clear that the following formulae hold:

$$F_l(x, y, \alpha x + \beta y + \gamma) = x^{n-l}(\alpha x + \gamma)^l \qquad (l = 0, 1, \cdots, n-1);$$

$$F_n(x, y, \alpha x + \beta y + \gamma) = \gamma^{n-1}y; \tag{4}$$

$$F_{n+1}(x, y, \alpha x + \beta y + \gamma) = (\alpha x + \gamma)^n + n\beta\gamma^{n-1}y.$$

Assume in particular that

$$\alpha \neq 0, \qquad \beta \neq 0, \qquad \gamma \neq 0.$$

Then the $n+2$ forms (2) and (3) are linearly independent; for the polynomials in x

$$x^{n-l}(\alpha x + \gamma)^l \qquad (l = 0, 1, \cdots, n-1)$$

are clearly linearly independent, since no two of them vanish to the same order at $x=0$ or at $x=-\gamma/\alpha$. Therefore the form of order n,

$$F(x, y, z) = \sum_{l=0}^{n+1} C_lF_l(x, y, z), \tag{5}$$

where $C_0, C_1, \cdots, C_{n+1}$ are arbitrary integers not all zero, does not vanish identically; it is again a solution of (1).

Take

$$\gamma = 1, \qquad C_n + C_{n+1}n\beta = 1$$

and choose F such that $F(x, 0, \alpha x+\gamma)$ is a polynomial in x exactly of degree n. Then by the remark at the beginning of this paragraph, $F(x, y, z)$ represents every integer in an infinity of different ways. We furthermore have

$$F(x, y, \alpha x + \beta y + \gamma) = F(x, 0, \alpha x + \gamma) + y,$$

and conclude that $F(x, y, z)$ is an irreducible form in x, y, z, since the expression on the right-hand side is irreducible in x and y and of exact degree n in x. It is again clear, as in the preceding paragraph, that the so constructed form, $F(x, y, z)$ is indefinite.

3. Forms with bounded greatest prime factor. A positive definite ternary form of order n, $F(x, y, z)$, represents every integer k in at most a finite number of ways. Since $F(x, y, z)$ is positive definite, its order is even, say $n=2m$. The values of F on the sphere $x^2+y^2+z^2=1$ are always positive; since F is continuous, they have a minimum value $V>0$ on this sphere. Hence

$$f(x, y, z) = \frac{F(x, y, z)}{(x^2 + y^2 + z^2)^m} \geqq V$$

for all points of this sphere, and therefore for all points of space, since $f(x, y, z)$ is homogeneous of order zero. Hence, if $k>0$ is given and $F=k$, then

$$k = F(x, y, z) \geqq V(x^2 + y^2 + z^2)^m, \quad i.e. \quad |x|, |y|, |z| \leqq (k/V)^{1/2m},$$

so that there are at most a finite number of integral solutions x, y, z.

Suppose in particular, that this form can be written as

$$F(x\ y\ z) = aQ(x, y)^m + (z - x - y)G(x, y, z), \tag{6}$$

where $a \neq 0$ is an integer and

$$Q(x, y) = \alpha x^2 + \beta xy + \gamma y^2$$

is a quadratic form in x and y, and $G(x, y, z)$ a form of order $n-1$ in x, y, z, both with integral coefficients. Since

$$F(x, y, x + y) = aQ(x, y)^m, \tag{7}$$

the form $Q(x, y)$ must be positive or negative definite; otherwise there would be real x, y, z not all zero such that $F(x, y, z)=0$.

By the theory of quadratic forms, it is possible to find for every integer $t \geqq 1$ a system of t different prime numbers

$$P_1, P_2, \cdots, P_t,$$

such that the equation

$$Q(x, y) = \mp P_1^{z_1} \cdots P_t^{z_t}$$

has an infinity of integral solutions $x, y, z_1, \cdots, z_t$, where x and y are relatively prime and $z_1 \geqq 0, \cdots, z_t \geqq 0$. Hence, by (7), there exist an infinity of different

sets of three relatively prime integers x, y, z, for which the greatest prime divisor of $F(x, y, z)$ is bounded.

It is possible to find positive definite forms $F(x, y, z)$ of every even order $n=2m$, which can be written as a sum (6). For instance, it suffices to take

$$G(x, y, z) = (z - x - y)H(x, y, z),$$

where $H(x, y, z)$ is a positive definite form of order $n-2=2(m-1)$. Or we may take $G(x, y, z)$ arbitrary, but such that $G(0, 0, 1)>0$, and then can make $F(x, y, z)$ positive definite by just choosing for a a sufficiently large positive integer.

In the excluded case that $Q(x, y)$ is indefinite, $F(x, y, z)$ evidently represents an infinity of different integers k in an infinity of different ways.

As there are many forms of the type (6), we may impose on them further conditions, *e.g.*, consider only forms which are quadratic or cubic forms in x^h, y^h, z^h. In the following examples, the sign "$\rightarrow$" means that the right-hand side is derived from the left-hand side by the substitution $z=x+y$.

(a) Quadratic forms in x^2, y^2, z^2. A few examples are given by the identities, in which α is arbitrary:

$$(1 - \alpha)x^4 + \alpha x^2y^2 + \alpha x^2z^2 + \alpha^2y^2z^2 \rightarrow (x^2 + \alpha xy + \alpha y^2)^2;$$

$$\alpha x^4 + \alpha(4\alpha - 1)y^4 + \alpha z^4 + (1 - 2\alpha)x^2z^2 \rightarrow (x^2 + xy + 2\alpha y^2)^2.$$

(b) Quadratic forms in x^3, y^3, z^3. The following identities hold for arbitrary α:

$$(\alpha^2 - 4\alpha + 4)(x^6 + y^6) + (\alpha^2 - \alpha + 1)z^6 + (3\alpha^3 - 16\alpha^2 + 28\alpha - 16)x^3y^3 - (2\alpha^2 - 5\alpha + 2)(x^3 + y^3)z^3 \rightarrow 3(x^2 + \alpha xy + y^2)^3;$$

$$(\alpha^2 - 3\alpha + 3)x^6 + \alpha^2(y^6 + z^6) + (2\alpha^2 - 3\alpha)x^3(y^3 - z^3) + (3\alpha^3 - 2\alpha^2)y^3z^3 \rightarrow 3(x^2 + \alpha xy + \alpha y^2)^3;$$

$$\alpha(x^6 + z^6) + (3\alpha^3 - 3\alpha^2 + \alpha)y^6 + (2\alpha - 3\alpha^2)y^3(x^3 - z^3) + (3 - 2\alpha)x^3z^3 \rightarrow 3(x^2 + xy + \alpha y^2)^3.$$

(c) A quadratic form in x^4, y^4, z^4:

$$x^8 + y^8 + 17z^8 + 14(x^4 + y^4)z^4 \rightarrow 2(2x^2 + 3xy + 2y^2)^4.$$

(d) There is no irreducible cubic form in x^4, y^4, z^4 with rational coefficients, but there are four with coefficients in $K(\sqrt{3})$ which are conjugate in pairs with respect to this field.

Final remark. Analogous to (6), there are positive definite quaternary forms $F(x, y, z, w)$ of every even order $n=2m$ which can be written as

$$F(x, y, z, w) = aQ(x, y, z)^m + (w - x - y - z)G(x, y, z, w),$$

where $a\neq 0$ is an integer, $Q(x, y, z)$ a positive definite quadratic form and $G(x, y, z, w)$ a form of order $n-1$. For forms F of this kind, the equation $F(x, y, z, w)=k$ has evidently at least const. $|k|^{1/n}$ solutions for an infinity of k's.

UNDERGRADUATE MATHEMATICAL RESEARCH*

F. L. GRIFFIN, Reed College

In returning to the scene of my own undergraduate work I find it a pleasant duty to acknowledge my deep indebtedness to that great college teacher, Herbert Ellsworth Slaught. The sense of exploring the unfamiliar, which permeated the initial two years spent under him, was a delightful experience. And he prepared students smoothly for the later years under Professors Moore, Bolza, and Maschke.

Spontaneous mathematical research arising out of the curiosity and reflections of inquiring and gifted young minds is no new phenomenon. Witness Pascal and Galois. More frequently a stimulating teacher has stirred an alert student to study some challenging question. Indeed, some institutions have long made a systematic effort to bring this type of experience to any student capable of profiting by it. In seeking to arouse interest in investigation they have utilized prize contests, departmental club programs, thesis requirements of candidates for honors, and so on. Many colleges now encourage a student to study independently some field new to him and to write a report of an expository or critical nature. From the standpoint of the student this may be called a type of research.

In the present discussion, however, I am using the word "research" in the more technical sense of exploring some new question or re-exploring some old question and getting results previously unfamiliar to specialists in the field. When a student investigates a question and gets results new to him but not actually involving priority, the experience may afford as valuable training for him as if he were the first to explore the matter. But in connection with the discussions of research under way in the present sessions of this Association and of the Society, it seems to me of possible interest to inquire how extensively research in the more technical sense is being carried on by undergraduates of our colleges.

It is, of course, a fair question how generally undergraduates ought to attempt mathematical research. For students who are going on to the graduate school, there is so much fundamental material to be learned as a basis for specialization that it is debatable whether time ought to be used for efforts in research. And whether a student is going on or not, one may wonder how much worthwhile research can be accomplished with the tools available in undergraduate days. Possibly the experience of institutions which have experimented with undergraduate research projects may throw some light on these questions.

Reports from various institutions. Thinking that one source of information might be the colleges and universities which according to published report†

* Read at the summer meeting of the Mathematical Association of America, Chicago, September 1, 1941.

† The American Council on Education, American Universities and Colleges, 1940.

require a thesis of some or all seniors, I sent a circular letter to the heads of the departments of mathematics in about ninety such institutions. The replies, not yet complete, run about as anticipated. Usually the senior thesis has the character of a critical expository essay or report. Seventeen institutions, however, report undergraduate research papers in the technical sense, arising either as theses or from individual initiative,—spontaneous combustion, as it were. The information thus far available runs as follows:

Albion College. In 1912 a paper on the normals to a conic embodied substantial research. Since 1930 research has been carried on extensively.

Beloit College. Two recent honors theses have involved research leading to new results. One dealt with the mathematical relations involved in a Precision Harmonograph; the other developed a variation in Newton's method of approximating the real roots of an equation.

Brooklyn College. The Mathematical Mirror, published annually for nine years by mathematics students of the college, has contained a number of research articles.

University of Buffalo. In the past twelve years there have been about 15 research papers, usually submitted in the Sherk Prize Contest sponsored by the Mathematics Club. Some papers have been presented at the New York State Undergraduate Scientific Congress.

University of California. Research is frequently done by students in going farther into problems which they have begun in some course. The topics have related to hyperspace, line geometry, synthetic projective geometry, higher plane curves, affine geometry, birational transformations, vector geometry and tensor theory, general postulate theory, Boolean algebra, real and complex variable, and problems found in Polya and Szegö's, *Aufgaben und Lehrsätze aus der Analysis*

University of Delaware. Honors theses, one or two each year, have dealt with such topics as semi-regular-continued fractions, Legendre's transformations, hypergeometric functions, and elliptic functions. New results of interest have been obtained in some cases.

Harvard University. For honors a thesis was long optional or required. Undergraduate research papers by A. L. Lowell and Maxime Bôcher were published back in the 1880's. Since 1926 a thesis has been required of students offering a concentration in any field. These theses often contain interesting results not in the literature. A recent thesis on the Chi-Square Tests has been published by the Harvard University Press.

University of Minnesota. For graduation *summa cum laude* a thesis is required. In mathematics a few theses have been of the research type.

University of New Mexico. For several years past a thesis has been required for graduation with honors. In mathematics about one thesis a year has made an original contribution. One of the latest developed a new implication relation in the calculus of propositions. Another studied the asymptotic behavior of certain functions of the Bessel type as the argument becomes infinite.

New York University. Allusion is made to a recent undergraduate publication.

Agricultural and Technical College of North Carolina. Since 1939 thesis work for students majoring in mathematics has been of a research character, directed through the Department of Physics. Abstracts of seven of these theses have been published in collaboration with members of that department.

University of Pennsylvania. In recent years a junior and a senior published important research papers on subjects in which they had become interested through advanced courses. One paper gave a new proof of the quadratic reciprocity law; the other, in the field of algebra, dealt with foundational questions.

Princeton University. Since 1927 a thesis has been required of each senior. The mathematical papers have frequently been of the research type.

St. Lawrence University. Two sophomores recently became interested in differentiators and constructed a machine.

University of Vermont. Since about 1890 a thesis has been required for graduation with honors. In the past ten years several of the papers have been of the research type.

University of Wisconsin. An exceptional student sometimes takes both the bachelor's and the master's degree at the end of four years. One outstanding thesis written by such a student studied the convergence of Newton's method of finding the roots of an algebraic equation when the roots, and possibly the coefficients, are complex numbers.

It would doubtless be of interest if the experience of the preceding institutions, and of others in which undergraduate research has been carried on, might be reported more fully by persons who have the necessary information.

Experience at Reed College. Please be indulgent if I now speak somewhat disproportionately about our experience at Reed College since I am most familiar with that. Since the beginning of the college a thesis has been required of all seniors in all departments, except students graduated in 1919. The thesis always involved some independent investigation; and after a few years, largely due to the interest of the students, it took on pretty generally a research character in the specialized sense. Of the 94 theses in mathematics written in the 26 years, at least 80 have involved research with some novel results; and in 56 of these I believe that the problem itself was new. At least, I have not been able to locate the problems in the literature.

Since a typical thesis, in a condensed form, has forty or fifty pages and would require about thirty minutes for an adequate description, perhaps the best I can do here is to suggest the range of topics treated and in a few instances allude to some results.

Analysis. The largest single group of our research theses falls in the Calculus of Variations, where of 17 theses fifteen have dealt with apparently new problems. Eight of these related to physical problems: *e.g.*, the form of the plane curved wire connecting two points and having minimum moment of inertia about an axis in its plane, or having minimum attraction for a particle; or carry-

ing an electric current through a wire of given length and producing the maximum magnetic force at a point; also, similar problems for surfaces of revolution. Seven other seniors determined the system of geodesics upon some surface, *e.g.*, the surface generated by revolving a hypocycloid of four cusps about a diameter through two cusps. In each case the student determined and plotted typical extremals, discussed their properties, studied the envelope of the one-parameter family through a single point and determined its singularities, if any; and in most cases gave a sufficiency proof for a relative extremum. Parametric equations involving elliptic integrals appeared in each problem, except two which required a graphical and numerical solution of Euler's differential equation.*

One thesis in this field interests me considerably, because it was written by a student whose work the first three years was of such low standing that we reluctantly accepted her as a senior. To our surprise she worked with unusual initiative and with application, and needed less guidance than some students of distinctly higher standing have had. She dealt with an old problem in an elementary way, "The geodesics on the torus," which was treated by a leading mathematician in both his master's and his doctor's theses. (While she was studying it I told her nothing of the earlier investigations, preferring, as advocated by Professor R. L. Moore, to let a student develop his or her own methods before consulting the literature.) She calculated the scales for conformal mapping of the torus and obtained and plotted the geodesics, using approximate integration extensively and accurately. I wonder whether it may be easier to do research of such a specialized type than to organize a broad field clearly, as for example our junior course in modern algebra which ranges from matrix theory to elements of the Galois theory.

Six other theses have discussed troublesome differential equations. A recent one worked with a system of equations relating to a hydraulically operated radial gate in a government dam at Eugene, Oregon. The motions were accurately calculated by numerical integration. This thesis, also, was written by a student whose previous standing had been only average, and whose energies had been devoted chiefly to actuarial mathematics. Another, written some years ago by a top-notch student, did a brilliant job in studying the effect of a shield on a coil carrying currents at radio frequencies. He set up a partial differential equation of order 2, similar to Maxwell's equations, got its solutions as series involving complex trigonometric and Bessel functions, discussed questions of accuracy, and got integral expressions for the back magnetic field and power loss. Still another paper on differential equations, perhaps the most mature thesis job we have had in our department was written this past year, with virtually no suggestions from an instructor. This dealt with a second-order equation which includes Bessel's and Riccati's equations as very special cases. Besides obtaining formal series solutions and discussing transformations, the stu-

* Interestingly enough, a student who wrote one of the latter theses got a job in the Ballistic Section, Bureau of Ordnance, where her work with trajectories has seemed almost a continuation of her thesis. One of her classmates had a similar carry-over in the Coast and Geodetic Survey.

dent defined a class of quasi-periodic functions and obtained 18 theorems relating these to the solution of his differential equation. (The same man in his freshman year had written a paper entirely on his own initiative in which he postulated two classes of elements, C and K, and two operations transforming an element of C into an element of K and vice versa, and deduced a series of theorems relating to repeated application of the operations.)

We have also had seven theses dealing with the calculation of elliptic integrals of type III. One of these used methods apparently not in the literature. The resulting tables, running to four decimal places, give values of $\pi(n, k, \phi)$ for a suitable range of values of n, k, and ϕ.

Two other research theses in analysis generalized the circular and hyperbolic functions of sectoral areas, modifying the circle and equilateral hyperbola or choosing the origin away from the center. Extensive systems of identities, derivatives, etc., were found.

Geometry. Turning to geometry, there is time to say only that we had a sequence of nine theses dealing with conjugate curves* and surfaces. In some of the classes studied, the conjugate of a line or conic is usually a higher plane curve, sometimes a familiar curve such as the cissoid, conchoid, strophoid, limaçon, etc., or perhaps another conic, or the polar of the origin with respect to a conic.

There is a group of four theses dealing with centro-surfaces of conicoids,† which are distinct from Cayley's center-surfaces. The centro-surface in some cases pinches down to a line and then expands again, so that certain plane sections may have nodes or cusps. The determination of the sections usually involves substantial algebraic difficulties.

There have been 15 other research theses in geometry, including generalizations of the conics in three dimensions, the inverse of the envelope problem, a study of the normals to an ellipse (which yielded fifteen new theorems), studies of constructibility by use of higher plane curves, and the projective determination of conics.

Algebra and Combinatory Analysis. In this field there have been five research theses, three of which obtained substantial and apparently new results. In one, criteria were obtained for the algebraic solvability of certain high-degree equations by reduction through a chain of reciprocal equations. In another, criteria as to the nature of the roots of the cubic and quartic were derived from Argand's representation of the roots of the resolvent equations. In 1916 a student who had been reading Ball's Mathematical Recreations and Essays decided to work on an unsolved case of "the schoolgirl problem" (or Euler's problem of the 36 officers): 36 girls are to walk in rows of 6; required, if possible, an arrangement by which each girl shall walk once and only once with each other girl. By an ingenious system of eliminating conflicts it was shown that the desired arrange-

* Defined in Bulletin American Mathematical Society, vol. 30, 1924, p. 15.

† Defined in Bulletin, American Mathematical Society, vol. 36, 1930, p. 483.

ment is impossible; and some related questions were treated. These results were reported to the Society, but were apparently overlooked (as was also a complete subsequent treatment of the general problem by Professor H. F. MacNeish)* when the same problem was treated some fifteen years later for a doctoral dissertation in one of our leading universities.

Further physical problems. Four theses have dealt with problems of light. Two of these have studied the effect of a line source of light, placed in certain positions, upon the performance of a parabolic mirror. Reflected rays diverging from the axis by more than arc sin .01 are considered as "lost"; and for various points on the luminous source, boundary curves are determined which separate regions of the mirror that "lose" light from regions that "save" light. The quantity lost is expressed by a triple integral of complicated form, which is approximated numerically.

Another thesis in this field studied the illumination in the courts of certain office buildings in Portland. By spherical trigonometry and descriptive geometry actual shadows, as contrasted with approximations often used, were constructed for the time of the equinoxes and the solstices. By using a photometer the variation of light in offices in a number of respects was studied.

One of our most original and spontaneous theses proposed a mathematical theory as to the nature of potential energy. The student, who while yet in high school had studied the theory of relativity, drew extensively upon modern physical theory. (He has since taken his doctorate in point-set theory which provides him with something of a contrast!)

Astronomy. A student, who for years had made variable star observations with a telescope of his own, analyzed these to determine the light curves of four stars. The usual methods for such analysis proved to be useless; but consistent patterns were obtained by choosing as the argument the ratio of the time interval since each maximum brightness to the length of the time in that particu'ar "cycle".

Economics. A thesis on a problem of duopoly extended a study of certain economic questions treated by Professor G. C. Evans.† Keeping his linear demand law, but replacing his quadratic cost function by a cubic, the student found various types of maxima occurring in the study of profits. A plaster model was constructed showing some peculiarities of the graphical surface in one case.

Statistics. A thesis submitted in 1930 dealt with lines of best fit, with special reference to the statistical determination of the demand curve for sugar. Three different basic assumptions as to the displacement of points in the scatter diagram from the required line were considered, especially the assumption that the displacements are normal to the line. Using numerical data for sugar, a detailed study was made of the surface which exhibits the sum of the squared deviations as a function of the parameters of the line. Systems of lines of "equally good fit"

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* Annals of Mathematics, 2nd ser., vol. 23, 1922, pp. 221–227.

† This MONTHLY, vol. 29, 1922, pp. 371–380.

were discussed, corresponding to points on contours of the surface; also, "indifference curves" whose tangents are such lines of equally good fit. (This thesis, supervised jointly by professors of mathematics and economics, took first prize in the Hart, Schaffner and Marx Economics Essay Competition for Undergraduates that year. Incidentally, it was written, with comparatively little guidance, by a student whose course work had been of only average quality.)

Some General Considerations. I have referred to some 56 of our Reed mathematical theses as having dealt with what appeared to be new problems. There are many such problems available, not sufficiently theoretical or advanced to justify their use for doctoral disserations but much too involved to permit their inclusion as exercises in textbooks. Others suggest themselves in considering possible modifications or generalizations of standard procedures or theories. Some of these theses were reported to the Society, due to lack of a section of the Association in our region.

While novelty is not essential for the student's profit in working upon a problem, it adds interest. The student greatly enjoys the experience of exploring new material, where the conclusions to be expected are not familiar to his instructor and are not availabe in the literature. In selecting a topic from a list of several possibilities the student may, if he likes, first glance over old theses to see what field or general type of investigation might suit him best. Sometimes he has a preference for a type of work very different from any of the suggested topics; and if he does not have a specific topic in mind an effort is made in consultation to formulate one.

One possible value of thesis work is that it gives the student an opportunity to make the most of whatever originality, independence, and spontaneity he may possess. It has long been my view that the tendency in undergraduate instruction is to neglect somewhat the most promising students in order to give adequate attention to the great middle group. In the charming town where I passed much of this past summer, one sees traffic signs near each school, which I suspect might be repunctuated thus as advice to the institution itself: "School, don't kill a child." In college, at least, it should not be necessary to kill the interest of our best students by limiting them to routine course work. Let me close with a quotation from a Reed College Bulletin on the subject of the senior thesis: "Latent powers of self-direction are developed in a more complete and extensive way than is afforded at any other point in the college career, and the student has the satisfaction of achieving a degree of mastery not commonly associated with the completion of a course."

THE ROOTS OF A QUATERNION

IVAN NIVEN, University of Illinois

The existence of an mth root of a quaternion α is known,* since the question reduces to the existence of a quaternion root of the equation $\xi^m - \alpha = 0$. We are led to inquire how many mth roots there are, and how to find them. The answer to the first inquiry is that there are exactly m distinct mth roots of a quaternion α which is not a real number; if α is real there are infinitely many mth roots unless $m = 2$ and α is positive, in which case there are only two square roots, $\pm\sqrt{\alpha}$. Our method of proof gives all roots.

LEMMA. *Any positive integral power of a quaternion $a+bi+cj+dij$ has coefficients of i, j, and ij which are proportional to b, c and d.*

The proof is by induction. Noting that the conclusion is true for the first power, we assume that

$$(a + bi + cj + dij)^{m-1} = A + q(bi + cj + dij).$$

Then we obtain at once

$$(a + bi + cj + dij)^m = aA - q(b^2 + c^2 + d^2) + (aq + A)(bi + cj + dij),$$

which proves the lemma.

Suppose now that α is any quaternion which is not a real number. Then the lemma enables us to say that any mth root of α must have the form $c\alpha+d$, c and d being real. The question is, simply, what real values of c and d can be used so that the equation

$$(c\xi + d)^m - \xi = 0, \tag{1}$$

has the quaternion α as a root? Now, designating the trace and norm of α by t and n respectively, we know that α satisfies the equation

$$\xi^2 - t\xi + n = 0. \tag{2}$$

This is the minimal equation with real coefficients satisfied by α, and hence our problem is to find real values c and d so that the left side of (1) is divisible by the left side of (2). The latter is the case if and only if the roots of (2) in complex numbers are also roots of (1). Since α is not real, the discriminant of (2) is negative, so that the complex roots of this equation can be denoted by the conjugates λ and $\bar{\lambda}$. Taking λ to be the one with positive imaginary coördinate, we can write

$$\lambda = r^m(\cos m\theta + i \sin m\theta), \qquad 0 < m\theta < \pi, \tag{3}$$

the m being inserted for convenience. We want to choose c and d so that

* Ivan Niven, Equations in quaternions, this MONTHLY, Vol. 48, 1941, pp. 654–661.

$$(4) \qquad (c\lambda + d)^m = \lambda, \qquad (c\bar{\lambda} + d)^m = \bar{\lambda}.$$

We can ignore the second of these two equations, it being a consequence of the first. Substituting (3) in the first part of (4) and taking mth roots, we obtain

$$(5) \quad cr^m(\cos m\theta + i \sin m\theta) + d = r\left\{\cos\left(\theta + \frac{2\pi k}{m}\right) + i \sin\left(\theta + \frac{2\pi k}{m}\right)\right\} \qquad (k = 0, 1, \cdots, m-1).$$

Taking conjugates of both sides of (5), and subtracting the result from (5), we get

$$(6) \qquad 2cr^m i \sin m\theta = 2ri \sin\left(\theta + \frac{2\pi k}{m}\right) \qquad (k = 0, 1, \cdots, m-1).$$

This can be solved for real c since $\sin m\theta \neq 0$, because of the inequality in (3). Substituting in (5) we obtain

$$(7) \qquad d = r \cos\left(\theta + \frac{2\pi k}{m}\right) - r \cot m\theta \sin\left(\theta + \frac{2\pi k}{m}\right) \qquad (k = 0, 1, \cdots, m-1).$$

We prove now that the mth roots of α obtained above are distinct. If the values of c in (6) are all different, clearly the quaternion values $c\alpha+d$ have the same property. If two values of c are equal, so that

$$\sin\left(\theta + \frac{2\pi h}{m}\right) = \sin\left(\theta + \frac{2\pi k}{m}\right), \qquad h \neq k,$$

then we can show that the corresponding values of d in (7) are unequal. For if they were equal, we would have

$$\cos\left(\theta + \frac{2\pi h}{m}\right) = \cos\left(\theta + \frac{2\pi k}{m}\right), \qquad h \neq k.$$

Since the angles involved here are unequal and lie between 0 and 2π, their sum must be 2π, that is,

$$\theta + \frac{2\pi h}{m} + \theta + \frac{2\pi k}{m} = 2\pi.$$

This simplifies to $m\theta = \pi(m-h-k)$, which contradicts the inequality in equation (3). We summarize our results.

THEOREM 1. *Given a quaternion α which is not real, solve the quadratic equation* (2) *in complex numbers (t and n being the trace and norm of α). Denote the modulus and amplitude of the root with positive imaginary part by r^m and $m\theta$. Then the m distinct mth roots of α are $c\alpha+d$, where c and d are given by equations* (6) *and* (7), *the same value of k being used simultaneously in the two equations.*

Henceforth let $\alpha \neq 0$ be real. Apart from real mth roots, any quaternion mth root of α satisfies a quadratic of the form (2), whose left side divides $\xi^m - \alpha$. Hence we search for real values t and n (these no longer denoting the trace and norm of α) so that this divisibility property holds; these values can be constructed as follows. If the real positive mth root of $|\alpha|$ be r, we may designate α by $r^m \cos s\pi$, s being 0 or 1 according as α is positive or negative. Then the complex roots of $\xi^m - \alpha = 0$ are

$$r\left(\cos\frac{(s\pi + 2k\pi)}{m} + i \sin\frac{(s\pi + 2k\pi)}{m}\right) \qquad (k = 0, 1, \cdots, m-1). \tag{8}$$

Ignoring any real values momentarily, we note that a conjugate pair of these are roots of (2) provided

$$t = 2r\cos\frac{(s\pi + 2k\pi)}{m}, \qquad n = r^2.$$

Since any non-real quaternion root of (2) must have trace t and norm n, it must have the form

$$r\cos\frac{(s\pi + 2k\pi)}{m} + yi + zj + wij \qquad (k = 0, 1, \cdots, m-1), \tag{9}$$

where y, z and w are any real values satisfying

$$y^2 + z^2 + w^2 = r^2 - r^2\cos^2\frac{(s\pi + 2k\pi)}{m} \qquad (k = 0, 1, \cdots, m-1). \tag{10}$$

Now among the values (8) there may be one or both of the real values r and $-r$. Note that any such values are also given by (9), since (10) implies that $y = z = w = 0$ in this case. Hence all quaternion mth roots of α are given by (9). There are infinitely many roots unless the right side of (10) is zero for every value of k, and this conditions obtains only if $s = 0$ and $m = 2$.

THEOREM 2. *Given any real number $\alpha \neq 0$, let r denote the real positive mth root of $|\alpha|$, and let s be 0 or 1 according as α is positive or negative. Then the mth quaternion roots of α are given by* (9) *with condition* (10) *imposed, the same value of k being used simultaneously in the two equations.*

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The Department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A COMMENT ON "DIFFERENTIALS"

DUNHAM JACKSON, University of Minnesota

The note on "Differentials" by Kac and Randolph (this MONTHLY, vol. 49, 1942, pp. 110–112) stresses the idea that when dy is defined in terms of dx the latter is to be regarded as an independent variable. May I suggest that the presentation would gain strength, not in its introductory stages but *after* the dust of construction and the smoke of discussion have been cleared away, by a summary for permanent reference such as the following, for which I make no claim to originality:

If y is a function of x, in symbols $y=f(x)$, *dy is a function of the two variables x and dx*, represented by the formula $dy=f'(x)dx$. Graphically dy is the increment of the ordinate of the tangent line, if x is given an increment Δx equal to dx. If dx is infinitesimal and $\Delta x=dx$ (and if each of the variables x, y has with respect to the other a derivative different from zero), dy differs from Δy by an infinitesimal of higher order (than dx or dy or Δy).

To this is to be added: The use of the differential notation in connection with the mutual dependence of three variables, any one of which is a function of either of the others (and similarly with a larger number of variables), is justified by, and serves to commemorate, the rule for differentiating a function of a function.

DIFFERENTIALS

ALONZO CHURCH, Princeton University

1. I am interested in the note of Kac and Randolph in the February number of the MONTHLY (pp. 110–112), because I agree with them that the usual definition of the differential which they criticize is unsound, and that this unsoundness is within the understanding of the more intelligent beginning student in the calculus—or at least that it is sufficiently near the threshold of his understanding so that the definition causes him difficulty (even if he cannot make explicit the reason for his difficulty).

I would urge, however, that the objection which they make to the usual definition is not sufficient to reveal an unsoundness in it. In effect their objection is that, in the equation

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

if dx is identified with Δx, the same variable Δx appears in the equation in two different rôles, on the left as a free variable and on the right as a bound variable.* But this must not be considered an error. For example, in a context where π has been defined as

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}},$$

no one would think of objecting to the equation,

$$\tan (x + \pi) = \tan x,$$

on the ground that π stands for an expression containing x as a bound variable. Likewise, in a context where $D_x y$ has been defined as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

there should be no objection to writing an equation which contains $D_x y$ and at the same time contains a separate occurrence of Δx as a free variable.

The modification of Kac and Randolph in the definition of the differential must therefore be considered as designed to remove a difficulty for the student, rather than as correcting an actual error. Its advantage is that it avoids the possible necessity for an added explanation, which would, at least in effect, have to reproduce the distinction between free and bound variables.

2. On the other hand, there is a more serious objection which applies alike to the definition of Kac and Randolph and to the more usual definition which they wish to replace.

This objection may be formulated in the following terms. Both definitions agree in defining the differential of the independent variable, say x, by taking dx to be a new independent variable—hence it should with equal correctness be possible to use a single letter, say z, to represent this new variable. Then the differential of a dependent variable, say y, is defined by taking dy to be $(D_x y)dx$ —*i.e.*, $(D_x y)z$. But a survey of the more usual purposes for which differentials are employed in the calculus will show that not all of these are adequately served by taking dx and dy to be z and $(D_x y)z$ respectively.

In particular, the use of differentials in connection with integration fails to be provided for. Thus $\int x dx$ becomes simply $\int xz$, and the whole significance of the notion is lost. (What plausibly is the operation $\int$ which, applied to the product of two independent variables x and z, yields $\frac{1}{2}x^2+C$?)—It should be emphasized that the facility which comes with the use of differentials is more marked in the integral calculus than in the differential calculus, and that any definition of differentials which fails to account for their use with the sign $\int$

* A variable is *free* in a given expression (in which it occurs) if the meaning or value of the expression depends upon determination of a value of the variable; in other words, if the expression can be considered as representing a function with that variable as argument. In the contrary case the variable is called a *bound* (or *apparent*, or *dummy*) variable.

has therefore lost more than half their value. Compare, *e.g.*, any one of the following processes in terms of differentials with the clumsier parallel process which regards the notation $\int \cdots dx$ as indivisible and employs only derivatives without differentials: (1) the integration $\int x\sqrt{4x+3}\,dx$ by the substitution $t^2=4x+3$, using the equation between differentials, $dx=\frac{1}{2}tdt$; (2) the solution of the differential equation $dy/dx=xy$ by multiplying both sides by dx/y and then applying the operation $\int$ to both sides; (3) the discovery by inspection of an integrating factor for a differential equation of the first order and first degree in two (or more) variables.

Another aspect of the foregoing objection lies in the point that the same notation, dx or dy, is given different meanings according as x is independent or dependent variable. This is especially unfortunate because it is often precisely one of the advantages in the use of differentials that various variables are symmetrically treated, without arbitrarily singling out one or more of them as the independent variable or variables (compare, *e.g.*, the equation $ds^2=dx^2+dy^2$ with the corresponding equation for $(ds/dx)^2$, or for $(ds/dy))^2$.

Sometimes a student will ask why the result,

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx},$$

cannot be obtained by simple cancellation of du against du, and the more difficult argument which employs properties of limits thus avoided. On the basis of the usual definition of a differential, the reply is to point out that du has different meanings in its two occurrences; but this immediately reveals the weakness of this usual definition.

If desired, the objection that the usual definition of a differential does not treat the variables symmetrically can be regarded as the fundamental objection. The operation $\int$ must, of course, be the inverse of the operation d, however the latter operation is defined; and if the operation d fails to treat the variables symmetrically, the same lack of symmetry must affect the inverse operation. The difficulty in connection with integration reduces in part to the point that the embarrassment occasioned by the lack of symmetry is more acute in the case of the inverse operation. But the matter is further complicated by the way in which the usual definition of dy introduces a new independent variable z.

Unless some solution of these difficulties can be found, it seems that it would be preferable to introduce differentials in a frankly inaccurate and heuristic manner as small values of the increment or "little bits" of the "variable quantity" involved, rather than to clothe the idea with the deceptive appearance of logical accuracy.

3. There is a method of introducing differentials which suggests itself as a possible remedy, but unfortunately it may not be suitable for use in an elementary course except by devoting a disproportionate amount of time to the

study of parametric equations. The statement of it which follows is at all events not intended to be in form for presentation to the student.

This method is simply to define dx and dy to be $D_\tau x$ and $D_\tau y$ respectively, where τ is an arbitrary parameter.*

This does not contradict the usual statement that differentials are direction numbers of the tangent line. On the contrary it implies that statement. But it also adds a supplement to it which is needed to provide for certain ordinary uses of differentials. In particular the sign $\int$ taken by itself is then naturally understood to mean integration with respect to τ.

The extension of this idea to the differential du of a function u of two independent variables x and y is possible but somewhat cumbrous. It would perhaps be preferable to interpret equations involving du as relative to an arbitrary functional relationship between x and y (*i.e.*, as holding for every such functional relationship which satisfies appropriate conditions).

No very convenient method is provided of introducing second differentials, or of associating differentials with small values of the increment. In fact it seems that, if this definition of differentials is adopted, the notion of a differential should be kept entirely separate from considerations connected with small values of the increment. A special notation to represent $f'(x)\Delta x$ may be unnecessary; if such a notation is introduced, it should be something like $\delta_x f(x)$ or $\delta_x y$ (the subscript being dropped only in cases where it is irrelevant which variable is independent).

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Mathematics in Agriculture. By R. V. McGee. New York, Prentice-Hall, Inc., 1942. 9+189 pages. $4.00.

This text covers many of the topics usually taught in a course in Agricultural Mathematics. The applications of mathematics to agriculture are illustrated by means of interesting, practical problems but this reviewer wonders whether problems of sufficient difficulty are included to challenge the student of exceptional ability.

* There should be no objection from the point of view of rigor to the introduction of an arbitrary parameter, as opposed to a particular parameter. It means that, in a more explicit formulation, function variables (or relation variables) would appear, corresponding to the fixed functions (or relations) which would be used in introducing a particular parameter.

The material in the first three chapters on operations, percentage and equations is of the degree of difficulty usually associated with high school algebra and arithmetic. The application of the material on lengths, areas, and volumes is evident to the agriculturist. The treatment of ratio and proportion is generous but variation is omitted entirely.

The discussion of the right triangle and trigonometry is reduced to the minimum essentials. The treatment of statistics consists of an extensive presentation of the arithmetic, geometric and harmonic means with applications. The term "standard deviation" is defined but very few problems on this topic are given. The chapter on graphs is well done and in the opinion of this reviewer is the best feature of the text.

The chapter on special applications of practical measurements contains much useful information. The author uses the mixture lever instead of the dairyman's rectangle in solving mixture problems. The treatment of exponents, logarithms, and slide rule is conventional. The inclusion of extensive tables, 23 pages, for future reference is a very desirable feature.

Conspicuous by their absence from this text the reviewer notices, for example, variation, progressions, interest and annuities.

FRED ROBERTSON

A Manual of Problems in Statistics. By S. Dayton. New York, Henry Holt and Company. 163 pages. $.95.

This work has been designed as a companion volume of problems to the customary undergraduate text in economic statistics (such as "Statistical Methods," by Frederick C. Mills). In this purpose, it has succeeded admirably. There is a wealth of problems which illustrate clearly the computational elements involved in virtually all the statistical methods in common use. The wide variety of topics handled include algebraic exercises, graphs, description of the frequency distribution, index numbers of prices and physical volume, analysis of time series (secular, seasonal and cyclical variation), simple correlation (grouped and ungrouped data and time series), multiple correlation, the normal curve of error, the Chi-square test, measures of reliability, small samples and analysis of variance. Two appendices give tables of problem data and the chief statistical tables. The data used in the problems are drawn from significant economic material and are presented in tables which are models of good construction.

A few more detailed remarks may be made on these technical aspects. Problem 2c in Section A involves the solution of three simultaneous linear equations; these equations are linearly dependent, which may confuse the student. The section on graphs, though covering a wide range, omits the Lorenz chart which has become prominent in recent years in connection with studies of income distribution. On the other hand, inclusion of the logarithmic standard deviation in describing frequency distributions and of the analysis of variance test for seasonal variation means treatment of two topics of importance which are frequently neglected.

However, some caveats may be entered on a more fundamental level. These may be illustrated by the problems given for describing frequency distributions, which involve calculation of several measures each of central tendency, dispersion and skewness. This disregards the logic of the mathematical theory of statistics (as developed by R. A. Fisher, H. Hotelling, J. Neyman and A. Wald), according to which these measures are used as estimates of parameters of the underlying probability distribution. There can, of course, be only one best estimate in each case. This same disregard of the mathematical assumptions underlying statistical techniques appears in the section on cyclical variation in time series. The essence of the proposed method (which is the standard one in statistics texts) is to divide the observed values by the product of the trend value and the seasonal index. However, the fitting of a trend by least squares involves the assumption that there is no cyclical variation. It might be objected that since there are no mathematically valid methods of dealing with cyclical variation, the suggested technique is at least a good approximation. Prima facie, this is a good argument; but it puts the matter up to practical experience. E. L. Frickey (see "The Problem of Secular Trend," Review of Economic Statistics, 1934) has shown that the average duration of a cycle as determined by standard methods from different trends fitted to pig iron production may range from 3.6 to 45 years (depending on the trend fitted). This would seem to suggest the unreliability of the method. Some attention might be paid to the simpler methods used by the National Bureau of Economic Research (see W. C. Mitchell and A. F. Burns, "The National Bureau's Measures of Cyclical Behavior," Bulletin 57 of the Bureau), which seem to have definite advantages over the standard method suggested by Mr. Dayton.

Two minor points may be disposed of before closing. In discussing correlation, use of the standard error of the correlation coefficient is suggested (Section J, problem 4), although the size of the sample is only 18; further, in using the formula to test the significance, the calculated value is substituted for r, instead of the theoretical value. In discussing the analysis of variance, the method is introduced (as is customary) by a comparison of two variances. The method of treating this problem (making the larger variance always the numerator in the variance ratio) amounts to using a 10 percent level of significance instead of the 5 percent level which is intended.

These remarks must not be interpreted to deny the great contribution this manual can make to the instruction of a statistics class. Most of the observations made on the relation of the material to the fundamental mathematical assumption are directed more against possible abuses by instructors than against the work itself. As a matter of fact, the sections dealing with measures of reliability, small samples and analysis of variance play close attention to the underlying statistical theory; and many recent developments have been incorporated in these sections. In fine, Mr. Dayton's work can be recommended as an invaluable aid in the teaching of elementary economic statistics.

KENNETH J. ARROW

Essentials of Trigonometry with Applications. By D. R. Curtiss and E. J. Moulton. New York and Boston, D. C. Heath and Company, 1942. 8+174+94 pages. $2.25.

The first five chapters of this book are identical with the corresponding chapters of the authors' Brief Course in Trigonometry, except for minor revision (Reviewed in this MONTHLY, vol. 47, page 560, 1940). The chapters on logarithms and on the solutions of plane triangles are well written. The former is commendably brief, without the sacrifice of any essential feature. The section on co-logarithms is starred for possible omission. In the chapter on the solutions of plane triangles the laws of sines and of cosines are proved without the use of projections.

In the part on spherical trigonometry, the right-angle triangle is first discussed, with emphasis on Napier's Rules. Again the laws of sines and of cosines are established without the use of projection. Napier's Analogies are then obtained as consequences of these. The applications include plane surveying, problems in artillery, plane sailing, including use of the unit mil in angular measurement. In spherical trigonometric applications, plane sailing, great circle sailing, determination of time of sunrise and sunset, and of positions on the Earth's surface, of azimuth and hour angle are included.

The tables are Heath's Logarithmic and Trigonometric Tables, prepared by Professor E. J. Oglesby. They include four-place tables of squares, trigonometric tables, radians, and logarithms of numbers. Then follow five-place tables of small angles, trigonometric functions, common and natural logarithms of numbers, and a selection of constants and their logarithms.

The printing and press-work are excellent.

VIRGIL SNYDER

The Principles of Financial and Statistical Mathematics. By Maximillian Philip. Revised Edition. New York, Prentice-Hall, Inc., 1941, 16+335 pages. $3.50.

This book is divided into three parts, Basic Mathematics, Financial Mathematics, and Statistical Methods. The first division contains material which the author believes is essential to students of financial and statistical mathematics. Many teachers will consider that most of the material in this part is too elementary for students who are mature enough for these subjects. The well presented topics on interest and logarithms are essential. The material pertaining to progressions, binomial theorem and approximate calculations should have been placed in this section of the text.

The second part contains subject matter usually presented in text books on financial mathematics, together with a short chapter on life insurance. Newton's method of interpolation, continuous decrease and increase and depreciation are well presented. The three methods of evaluating the price to be paid for a bond are of much interest and should prove helpful to the reader. Students who need the knowledge of the mathematics in most of Part I before proceeding to the

second and third parts will find it extremely difficult to follow what is presented about logarithmic series, the base of natural logarithms and continuous increase and decrease; others should appreciate these ideas.

The problems in this part should help one to apply the methods presented and should furnish the teacher with adequate material for assignments.

The last part contains subject matter pertaining to statistics. The presentations of the normal curve from a ratio and the straight line in relation to different kinds of plotting paper are good. The development of the method of least squares by minimizing a quadratic expression will enable those who have not studied the calculus to understand and appreciate this method. Very little use is made of probability as applied to statistical problems. Many important topics which should be introduced in a beginning course in statistics are omitted. There is a shortage of good problems in this part. Little space has been given to interpretations of results. It is a mistake to omit ideas concerning sampling theory from any test on statistics.

The tables which accompany this book contain square and cube roots, common and natural logarithms, $(1+i)^n$, $(1+i)^{-n}$, $s_{\overline{n}|i}$, $a_{\overline{n}|i}$, $1 \div a_{\overline{n}|i}$, $1 \div s_{\overline{n}|i}$, $(1+i)^{1/p}$, log $(1+i)$, bond table, e^x, e^{-x}, life expectation, mortality table, net and gross annual premiums, ordinate and areas for the normal curve, log $(n!)$. These tables are well arranged, and are very easy to read.

W. D. BATEN

A Mathematician's Apology. By G. H. Hardy. Cambridge, The University Press; New York, The Macmillan Company, 1940. 7+93 pages. $1.00.

Here is a little book every mathematician should have in his circulating library. This book should be read and then loaned; first, to every young person with mathematical leaning, then to non-mathematicians, and finally to any mathematician who inadvertently has failed to buy a copy.

The young mathematician will learn, probably to his surprise, that his very youth with its freshness and lack of restraint, is a much greater asset in his chosen field than the greater knowledge and experience that he hopes to accumulate over the years to come. He should realize after reading this book how important it is for him to pour what he has into an all out effort at the very beginning of his carrer. At that time, according to Hardy, he is more likely to win fame and place (but not fortune) for himself than at any later date. An old mathematician, as an old tennis player, may be a fairly young man.

A non-mathematician may be well repaid for two of his hours spent with this book, but he would profit relatively more if he invested five or six. Even the person capable of the remark "Who cares if there is an infinity of prime numbers?" may feel a sense of satisfaction after Hardy has led him to this trough and made him drink. Of course the non-mathematician will puzzle over the sentence "Real mathematics has no effect on war," but it will be good for his soul to be jarred from his notion that school mathematics is the ultimate in

mathematics. He will, of course, not see what lies beyond but must realize that there is something to be seen.

A mathematician needs no apology for mathematics, but he can read this book and after every section feel a deep sense of gratitude to one of his greatest contemporaries for putting the cause so clearly and with such force. He may not have had the nerve to say for himself, but he will be moved when he reads "and there are probably more people really interested in mathematics than in music."

It is not to be expected that all mathematicians will agree with Hardy on every point. There may, for example, be mathematicians who object to Hardy's use of the term "trivial mathematics," and certainly there are "real mathematicians" who would not feel they were criticizing the bible if they threw a little mud at some "real mathematics."

This book is not only about mathematics, it is about ideals, art, beauty, importance, significance, seriousness, generality, depth, young men, old men and G. H. Hardy. It is a book to be read, thought about, talked about, criticized, and read again.

J. F. RANDOLPH

The College Placement Algebra Workbook; *Defense Mathematics*. By A. E. Fulton. Marietta, Georgia, Kennesaw Publishing Co., 1942. 112 pages. $.75.

This little book is a series of tests of from four to some twenty exercises, on perforated sheets to be removed as used. Preceding each test is a short summary of the processes to be employed, largely reducing it to a mechanical procedure. These begin with the most elementary processes of simplification, addition, subtraction, multiplication, division, and factoring. The statements are given without proofs or motivation. When a new symbol is introduced, the practice is confusing, for example, radicals, p. 27. The introduction of imaginaries is at least naive. These and graphs are so brief as to be really harmful. The same statement applies to a solution of a system of simultaneous linear equations, p. 39.

To this early list of tests is added a considerable selection from actual examination papers given at the Naval Academy, without the summary of the processes involved.

In addition to the incorrect and confusing statements made throughout the book, there is also a persistent suggestion of the advertising purpose of the course. This is probably not more objectionable than in many other cases, but is a lamentable tendency under the present circumstances.

On the whole the above remarks are applicable to a considerable literature on short cuts in mathematical teaching. They are not the opinions of one teacher, but the result of a consensus of opinions of a considerable number of teachers.

VIRGIL SNYDER

NEW BOOKS RECEIVED

Essentials of Trigonometry with Applications. By D. R. Curtiss and E. J. Moulton. New York and Boston, D. C. Heath and Company, 1942. 8+174+94 (Tables) pages. $2.25.

Mathematics in Agriculture. By R. V. McGee. New York, Prentice-Hall, Inc., 1941. 9+189 pages. $4.00.

Tables of the Moment of Inertia and Section Modulus of Ordinary Angle, Channels and Bulb Angle with Certain Plate Combinations. New York, Work Projects Administration, 1941. 13+197 pages. $1.25.

Intermediate Algebra for College Students. By T. S. Peterson. New York and London, Harper and Brothers, 1942. 8+358 pages. $1.85.

Technidata Hand Book. By E. L. Page (Essential Data on Mathematics, Physics, Chemistry, Engineering, Mechanics. For Engineers, Designers, Chemists, Mechanics and Technical Students.) New York, Norman W. Henley Publishing Company, 1942. 64 pages. $1.00.

Exploring Numbers. By H. G. Campbell and F. L. Wren. New York, D. C. Heath and Company, 1942. 7+264 pages. $0.80.

Number Activities. By H. G. Campbell and F. L. Wren. New York, D. C. Heath and Company, 1942. 7+247 pages. $0.80.

Number Experiences. By H. G. Campbell and F. L. Wren. New York, D. C. Heath and Company, 1942. 7+248 pages. $0.80.

Discovering Numbers, by H. G. Campbell and F. L. Wren. New York, D. C. Heath and Company, 1942. 7+280 pages. $0.80.

Supplement to Pandiagonal Magic Squares of Prime Order. By A. L. Candy. Lincoln, Nebraska, A. L. Candy, 1942. 1+30 pages.

The College Placement Algebra Workbook; *Defense Mathematics*. By A. E. Fulton. Marietta, Georgia, Kennesaw Publishing Company, 1942. 112 pages. $.75.

Operational Methods in Applied Mathematics. By H. S. Carslaw and J. C. Jaeger, Oxford, University Press, 1941. 16+264 pages. $5.00.

Mathematics in Daily Use. By W. W. Hart, C. Gregory and V. Schult. Boston, D. C. Heath and Company, 1942. 8+376 pages. $1.32.

Portraits of Famous Physicists with Biographical Accounts. By H. Crew. New York, Scripta Mathematica, 1942. 12 Portraits.

Calculus. By A. L. Nelson, K. W. Folley, and W. M. Borgman, Boston, D. C. Heath and Co., 1942. 10+356 pages. $2.75.

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT AND J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Brown University, Providence, R. I.

THE FILM—A TRIPLE INTEGRAL

E. A. WHITMAN, Carnegie Institute of Technology

In a recent leaflet entitled *Library of the American Mathematical Society*, Professor R. C. Archibald, the Society's long time, enthusiastic, and efficient Librarian said "In the near future films are sure to become very important library adjuncts." Already there are extensive film libraries on many subjects but the number of films on mathematics is distinctly limited. As the National Council of Teachers of Mathematics program on Visual Aids at the recent mathematics meetings at Bethlehem, Pennsylvania, revealed several new films, we may suspect that a considerable number of teachers of mathematics are considering the possibilities of films. Yet the experiences of these are seldom finding their way into print. The belief that an exchange of experiences with films might be desirable leads this writer to describe briefly the production of the film *A Triple Integral*. In addition, the showing of this film at our recent Christmas meetings was followed by inquiries which seemed to indicate interest in the details of its production.

This film resulted from an experiment in the use of the animated picture idea as an aid in class room teaching. There has been no attempt to include any motivating effects. Triple integration is one topic where it seems that the process could especially well be illustrated by a large number of drawings, too many for the teacher to make before a class. For example, suppose a solid has been shown on the blackboard and the *element of volume* selected, say a rectangular parallelepiped with edges Δx, Δy, and Δz. Now the question arises as to what happens if other solids like our chosen element of volume are piled one on top of another to form a *column* parallel to the z-axis. Thus far the work can be shown by drawings on the blackboard or even better yet by supplementing the drawings by use of a model. But the calculus requires the limit of the sum of the elements of volume as Δz approaches zero, Δx and Δy remaining constant. The number of elements of volume is increasing and the height of the column is changing. Here the large number of drawings required to sketch the process makes the task too long for the class room even for those teachers who can do a good job of sketching. Here it would seem that the animated picture might well be used to show the process. When the columns are to be summed to form a *slice* and the limit of this sum is considered, the figures become increasingly complex. At the third integration when the slices are summed the situation is still more complicated. Yet all the figures can be shown in a very short time by the use of a film.

In making the film *A Triple Integral* about five hundred twenty drawings and some forty titles were used. These drawings represented only about one fourth of the drawings made since in developing this topic the process was necessarily one of trial with no previous film to give guidance. The drawings were made by students as a part of a National Youth Administration project and required some six hundred student hours. The work however has been good training both in learning something of the meaning of precision in drawing as well as in preparation for the calculus. The photography in this the fourth trial took some twenty hours. The drawings were made on thin vellum paper instead of the celluloid sheets used commercially. The latter are said to require greater technical skill in drawing. The photography was accomplished by placing the drawings and titles on opal glass with light beneath and camera above. Our camera could take single exposures as well as run continuously. It did not have a rewind feature so it was not possible to make those easy changes between successive titles or between the greater changes in the development of the story. The animation was accomplished by placing different drawings over various constant backgrounds but the character of the film seemed to require much slower animation than commercial movies. The showing time for titles required some experimentation. The common rules for titles anticipate only the needs of very juvenile minds.

The technical difficulties in accomplishing animation are a real stumbling block in this type of work, at least so our experience goes. Our paper drawings would stretch. Our system of indicating where the drawings were to be placed with reference to their backgrounds was not sufficiently accurate to prevent undesirable internal motions. Even this fourth attempt at filming has produced something that one shows only with hesitation. It would seem to be an ideal situation if the mathematician could do the development work and the final set of drawings and the photography could be done in some studio that was equipped and manned for such work. After all our mathematics text books owe much to the art of the printer. Might we not expect then that fine or even acceptable films might require the skill and art of the movie studio?

Such studio production would seem to need either a wealthy patron or an interested sponsor. Sometime the question of sponsoring films will likely come before our mathematics organizations. Then there is also the possibility that some manufacturer may sponsor some films in order to increase the need for equipment in somewhat the same way that manufacturers of radio receiving sets have sponsored radio broadcasts. Studio production would make films generally available, and only under such conditions of availability can films have wide enough circulation to take them out of the hobby class.

Meanwhile it may be desirable to promote the exchange of experiences between those interested in trying to produce films in the field of mathematics. To this end the film *A Triple Integral* will be gladly sent to any one interested in it. To those who have inquired about having it generally available, the only reply at present is that this must await future developments.

CLUB REPORTS, 1941–1942

Pi Mu Epsilon, University of Oklahoma

Expanding horizons was the topic of Dr. J. O. Hassler and Balfour Whitney at the first meeting of the chapter held at the university observatory. Other programs were devoted to a discussion of *Arabic mathematics, yesterday and today* by Richard Dulaney who spent two years in Arabia with an engineering company, and to *Elementary groups* by Stuart Lee. The annual mathematics contest sponsored by the society was won by Abdurrahman Durakel who had been sent to study at the university by the Turkish government. Mr. Durakel was presented with Courant's *Differential and Integral Calculus*. Professor O. H. Hamilton of Oklahoma Agricultural and Mechanical College was guest speaker at the banquet and initiation and spoke on *Mathematics as an avocation*. Part of the entertainment at the banquet was furnished by problems and puzzles which appeared on the printed program. Officers were: Director, Harold Huneke; Vice-Director, Stuart Lee; Secretary, Marian Wright; Treasurer, Phyllis Barclay; Sponsor, Dr. Dora McFarland.

Delta Zero Mathematics Club, Mississippi Delta State Teachers College

This club was organized in September 1940 under the sponsorship of Dr. J. A. Ward. A constitution with by-laws was composed and approved. Topics for programs held during the year were: *The use of matrices in secret codes*; *Five solutions of an equation—Factoring, completing the square, formula, geometric drawing and slide rule*; *Famous men in mathematics—Newton, Archimedes, Pythagoras*; *Interesting facts about fourth dimensional cubes* (with models); *Three ways of trisecting angles*; *Three ways to take the square root*; *Speed and force of a bullet shot from different angles*; *Aeronautical movies*, showing the connection between mathematics and aviation. Officers were: President, William MacDonald; Vice-President, Christine Douglas; Secretary-Treasurer, Connie Scott; Reporter, Ranny Williams.

Pi Mu Epsilon, Oregon State College

Subjects and speakers at the six meetings held during the year were: *Works of Gauss* by Dr. Henri Scheffe, *Duo-decimal system* by Jean King, *Graphical solutions of equations* by Karl Steinbrugge, *Cantor's theory of trans-finite numbers* by Dr. Andrew Sobczyk, *The isograph* by Calvin Gross, *Numerical solution of a pursuit problem* by Robert Beagles, *Heaviside's unit function* by Helen Murdock, *The mathematical relationships of the frequencies of musical tones in scales and chords* by Eugene Grant. The chapter awards a scholarship annually to a student registered in mathematics and also a yearly prize in mathematics to the Willamette Valley Science Conference. Director, Eugene Grant; Vice-Director, J. Peterson; Secretary, Annabelle Berg; Treasurer, Professor G. A. Williams.

Mathematics Club, Women's College, University of Delaware

The first meeting was devoted to a discussion of the development of the old Greek curves of double motion. Topics and speakers at later meetings were: *Applications of mathematics to astronomy* by Professor R. W. Jones, *Mathematical Logic* by Ann Hamilton, *Types of proof of the Pythagorean theorem with illustrations* by Dr. G. H. Wilson of the Department of Physics. At the annual banquet Dr. C. J. Rees discussed five equivalent definitions of the sine function. President, Grace Shockley; Vice-President, Katherine Mitchell; Secretary-Treasurer, Alice Ward; Faculty Adviser, Edith A. McDougle.

Mathematics Club, Brown University

This club held its usual series of six monthly meetings from November to April. These were planned in advance and announced on a printed program. Undergraduate speakers at four of the meetings included Ellen Swanson on *Coincidences and probability*, Harvey Spear on *Election by proportional representation*, Donald Hall on *One sided surfaces*, Albert Acorn on *The bridges of Koenigs-*

berg, Tamara Bachman on *Curiosities in numbers*, Paul Tamarkin on *Minimal surfaces and soap films*, Arthur Long on *Euler Phi-function* and Herbert Maass, Jr. on *Mathematics in aeronautics*. Faculty speakers included Professor J. A. Clarkson of the University of Pennsylvania who spoke on *Curves of constant width*, and Professor J. S. Frame who demonstrated and discussed *Stencils for detecting primes*. Each year a picture is taken of the club and added to a collection of such pictures which date, with a few interruptions, back to the founding of the club in 1915.

Pi Mu Epsilon, University of Missouri

Activities for the year included five program meetings, a picnic, social meeting in which mathematical tricks and puzzles were the source of the evening's entertainment, and an initiation banquet at which 26 new members were admitted. Topics discussed were: *Dimensionality—three and otherwise* by Dr. Herman Betz, *Some applications of statics to geometry* by L. M. Kelly, *Mechanical description of curves* by Olen Nance, *Hyperbolic functions* by Dr. B. E. Gillam, *Elementary concepts from number theory* by W. E. Ferguson. Director, D. F. Abell; Vice-Director, G. E. Brown; Corresponding Secretary Dr. L. M. Blumenthal; Treasurer, Dr. B. E. Gillam.

Mathematics Club, University of Cincinnati

Articles from the MONTHLY were used for discussion at several of the meetings. *Soap films and minimal surfaces* were discussed by Adolph Goodman and Ruth O'Donnell. *Wine carrying and river crossing in a general case* was the topic of Robert Buck. Other titles were: *Pigs is pigs—a discussion of three standard problems of the Diophantine type* by Everett Yowell, *Polynomials simultaneously orthogonal on two or more circles* by Stanley Lawwill, *The number theory of quaternions* by Earl Swafford, *Regular polyhedra* by Robert Buck, *Symmetry* by Keyser Kunz, *Measurement in education* by Harvey Weitkamp, *The un-necessity of the ruler* by Ruth O'Donnell, *Elementary functional equations* by W. E. Restmeyer. President, Adolph Goodman; Vice-President, Robert Buck; Secretary-Treasurer, Mary Connor.

Mathematics Club, Hunter College of the City of New York

During the week set aside for dedication ceremonies for the new Hunter College Building, the club sponsored a Bureau of Vital Statistics. They showed pictorially the various data on the new building, the students and activities pursued. Programs included the following papers: *Alternating current developed by means of complex numbers* by Frieda Axelrod, *Invariants in the analytic geometry* by Professor C. C. MacDuffee, *P-adic numbers* by Shirley Orlinoff, *Transformations* by Professor Jewell Hughes Bushey, *Mathematical tricks, puzzles, and fallacies* by Dora Feldman, *Darkening at the limb of the sun* by Tecla Combariati, *Propositional functions* by Louise Miller, *Undergraduate mathematical publications* by Professor Lao G. Simons. President, Tecla Combariati; Vice-President, Charlotte Gertler; Secretary, Phyllis Monderer; Treasurer, Hortense Schindler; Faculty Adviser, Dr. L. A. Aroian.

Harvard Mathematical Club, Harvard University

Speakers and their topics for the year included: *The principle of sufficient reason* by Professor G. D. Birkhoff, *Introduction to the prime number theorem* by the Rev. R. E. O'Connor, *Parallelism, geodesics and related topics* by Caro Lippman, *Milne's kinematical relativity* by John Breakwell, *Some curious properties of numbers* by Dr. A. Whiteman, *Transfinite arithmetic* by Professor Garrett Birkhoff, *The elusive Plücker characters* by Professor Oscar Zariski, *The Fibonacci sequence* by E. C. Gras, *Radix and number systems* by G. F. Forbes, *Some questions in analysis* by H. Pollard, *An elementary fixed point theorem* by Professor Alfred Tarski, *Bertrand's postulate* by C. Price, *Some paradoxes in measure theory*, by J. Eisenstein, *Is it worthwhile to study mathematics?* by Professor George Sarton. Radcliffe College students are admitted as associate members of the club. President, E. C. Gras; Vice-President, Murray Lampert; Secretary, Julian Eisenstein; Treasurer, Alfred Putnam; Faculty Adviser, Professor D. V. Widder.

Pi Mu Epsilon, Columbia University

At the beginning of the year, the committee on reorganization which had been elected at the close of the previous year, sent out letters to a select group of students. Replies were received from over twenty five and seventeen of these were ultimately inducted into the society during the year. Bi-weekly meetings were held at which the following subjects were presented: *The mathematical theory of gases* by Professor L. P. Siceloff, *Mathematics and semantics* by Leon Henkin, *Contour integration* by Lawrence Annenberg, *Dimensions* by Professor P. H. Smith, *Tschirnhaus transformations* by Richard Brown, *The summation of series* by Bernard Gelbaum, *Transfinite numbers* by Kenneth Miller, *Mathematics and economics* by Harry Schwartz, *Hyperbolic geometry* by Dr. W. Strodt. Additional meetings were also devoted to the solution of problems and preparation for the Putnam and the Metropolitan New York *Pi Mu Epsilon* contest. Director, Leon Henkin; Secretary-Treasurer, Ulrich Strauss; Faculty Adviser, Professor L. P. Siceloff.

Mathematics Club, The University of Kansas

Semi-monthly meetings were held throughout the year. Printed programs were issued at the beginning of the year and papers were presented as follows: *The foundations of mathematics* by Bruce Crabtree, *Numbers* by Kenneth Barnett, *The first quadrature of a curvilinear surface* by Clark Moots, *Descartes and the invention of analytic geometry* by Arthur Ames, *Some famous problems of modern mathematics* by Professor G. B. Price, *Newton, Leibnitz and the invention of the calculus* by John Tweed, *Limits* by Merle DeMoss, *The theory of numbers* by Calvin Foreman, *Probability* by L. R. Shobe, *A problem in biophysics* by Wellesley Dodds, *Non-Euclidean geometry* by Ralph Burson, *The Lorentz transformation* by Arthur Peters, *Iteration method of solving equations* by Professor H. E. Jordon, *Generalized electrical equations and the M.K.S. system* by Professor Hessler. President, Bruce Crabtree; Vice President, Merle DeMoss; Secretary-Treasurer, Kenneth Barnett; Faculty Adviser, Professor G. B. Price.

Mathematics Club, University of Rochester

This club was reorganized early in the year after a lapse of several years. A quiz program in the nature of an Information Please contest was conducted at a mid-year meeting. Papers presented at the remaining meetings included: *The flow of heat and the application of Fourier analysis* by David Van Horne, *Quadratic Diophantine equations* by Robert Mann, *Linear Diophantine equations—the problem of the monkey and cocoanuts—and similar problems* by Helen Nyquist, *Nomograms and their uses* by George Monroe, *The Minkowski geometry of numbers* by David Falkoff. President, Robert Mann; Secretary-Treasurer, Catherine Reid.

Pi Mu Epsilon, University of Arkansas

During the year, this chapter held seven regular meetings, two picnics and two initiation banquets. At the latter, twenty six new members were admitted to the organization. Papers were given as follows: *Gambling* by Howard Head, *Proof that the proverbial "dog did catch the rabbit"* by Robert Morse, *Apportionment of representatives in Congress* by Landon Brown. A mathematics contest was held at one of the banquets. President, John Turner; Vice President, Bobbie Alfrey; Secretary, Robert Hobson; Treasurer, Harry Clayton; Faculty Adviser, Dr. V. W. Adkisson.

Mathematics Club, Wayne University

At the semi-monthly meetings held during the year the following papers were discussed: *Configuration symbols in plane projective geometry* by Jack Swartz, *The use of latent squares in finding regression lines* by Clifford Simms, *The theorem: If any odd perfect number exists, it must have at least six prime factors* by Robert Coveyeau, *Trisection of angles by various means* by Jack Swartz, *The simplification of the equation of the conic* by Eva Rossman. President, Jack Swartz: Vice-President, Rollin Woodward; Secretary, Betty Pickering; Faculty Adviser, Dr. D. C. Morrow.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 526. *Proposed by R. C. Yates, Louisiana State University*

Find the locus of P if the angles formed by the tangents from P to two fixed circles are equal.

E 527. *Proposed by V. Thébault, San Sebastián, Spain*

Show that the sum of the radii of the circles C_1, C_2, C_3 of E 457 [1941, 637] is equal to the diameter of the incircle, and that the sum of the radii of the three analogous circles whose centers are exterior to the segments A_iI is three times as great.

E 528. *Proposed by R. A. Rosenbaum, Reed College*

Prove and generalize the identity

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k^2}\cdot e^{-k}=e^{-1/2}.$$

E 529. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Construct an irregular hexagon which shall be both inscriptible and circumscriptible.

E 530. *Proposed by P. D. Thomas, Southeastern State College, Okla.*

Is there a sphere orthogonal to the six radical spheres determined by four given spheres whose centers are not coplanar? (The radical sphere of two spheres is the locus of a point whose two powers have zero sum.)

SOLUTIONS

The ith Root of -1

E 491 [1941, 635]. *Proposed by E. T. Frankel, Albany, N. Y.*

Prove that $\sqrt[\sqrt{-1}]{-1}=23\frac{1}{7}$, approximately.

Solution by I. Opatowski, University of Minnesota

The ith root of the complex number $z=e^{\theta i}=e^{(\theta+2k\pi)i}$ is $e^{\theta+2k\pi}$ where k is an arbitrary integer. When $z=-1$, this becomes

$$\sqrt[i]{-1} = e^{(1+2k)\pi}.$$

Since $e^{\pi} = 23.14069 \cdots$, the value sought by the proposer is that corresponding to $k = 0$.

Also solved by R. K. Allen, Aaron Bakst, W. B. Brown, R. E. Crane, William Douglas, Thor Eriksson, Howard Eves, Edward Fleisher, G. S. Heller, J. F. Kenney, C. N. Mills, C. C. Oursler, P. W. A. Raine, W. L. Roberts, E. P. Starke, Alan Wayne, and the proposer.

Four- and Six-digit Squares

E 492 [1941, 635]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a four-digit square which remains a square when two zeros are intercalated between the thousands digit and the hundreds digit.

Solution by Alan Wayne, Rhodes School, New York

From Barlow's Tables, there are only seven 6-digit squares whose second and third digits are zeros. These lead at once to two 4-digit squares satisfying the conditions, namely

$$2704 = 52^2 \qquad (200704 = 448^2),$$

$$7569 = 87^2 \qquad (700569 = 837^2).$$

Also solved by W. E. Buker, M. L. Constable, Thor Eriksson, Evelyn Hesseltine, E. P. Starke, and the proposer. (The larger solution alone was found by D. H. Browne and William Douglas.)

A Tetrahedron and Four Spheres

E 492 [1941, 635]. *Proposed by N. A. Court, University of Oklahoma*

Given a tetrahedron $ABCD$ and a point M, prove that the tangent planes, at M, to the four spheres $MBCD$, $MCDA$, $MDAB$, $MABC$, meet the respective faces BCD, CDA, DAB, ABC in four coplanar lines.

Solution by Howard Eves, Allen Academy, Bryan, Texas

Given a tetrahedron $ABCD$ and a point M, we can easily prove that the planes through M, parallel to the faces BCD, CDA, DAB, ABC, meet the respective spheres $MBCD$, $MCDA$, $MDAB$, $MABC$ in four cospherical circles. For the perpendiculars to the four circles, at their centers, coincide with the perpendiculars to the faces BCD, CDA, DAB, ABC at their circumcenters. Since these concur (at the circumcenter of the tetrahedron), it follows that the four circles (through M) are cospherical.

The desired theorem can be derived from this by inversion, with center M.

Also solved by the proposer.

A Diophantine Equation

E 494 [1941, 635]. *Proposed by Henry Scheffé, Reed College*

Show that, for every positive integer N, the equation

$$x^2 + y^2 + 2xy - 3x - y + 2 = 2N$$

has a unique solution in positive integers x, y. Give a method for finding it, without successive trials. Generalize the problem to n unknowns $x_1, x_2, \cdots, x_n$, satisfying

$$F(x_1, x_2, \cdots, x_n) = 2^{\alpha-1} N,$$

where F is a special polynomial of degree $\alpha = 2^{n-1}$, with integral coefficients.

Solution by W. B. Carver, Cornell University

The equation

$$x^2 + y^2 + 2xy - 3x - y + 2 = 2N \tag{1}$$

may be written as $\binom{t}{2}+y=N$, where $t=x+y-1$. Hence all its solutions in integers are given by

$$x = \binom{t+1}{2} - N + 1, \qquad y = N - \binom{t}{2}, \tag{2}$$

where t is an arbitrary integer. If now we require that x and y be *positive* integers, we must have

$$\binom{t}{2} < N \leqq \binom{t+1}{2}.$$

For a given positive integer N, this defines a unique positive integer t, namely the greatest t for which $\binom{t}{2} < N$. Substituting this in (2), we have the unique solution of (1) in positive integers.

To generalize the problem, we define a function of n variables by the recurrence formula

$$f_i(x_1, \cdots, x_i) = f(f_{i-1}(x_1, \cdots, x_{i-1}), x_i), \qquad (i = 3, 4, \cdots, n), \tag{3}$$

where $f(x, y) = f_2(x, y) = \frac{1}{2}(x^2+y^2+2xy-3x-y+2)$.

It is readily seen that the function

$$F(x_1, \cdots, x_n) = 2^{\alpha-1} f_n(x_1, \cdots, x_n) \tag{4}$$

is a polynomial of degree $\alpha = 2^{n-1}$ with integral coefficients. Writing the diophantine equation

$$F(x_1, \cdots, x_n) = 2^{\alpha-1} N \tag{5}$$

as

$$f(f_{n-1}(x_1, \cdots, x_{n-1}), x_n) = N,$$

we see that it is satisfied by unique positive integral values of f_{n-1} and x_n, say $f_{n-1}(x_1, \cdots, x_{n-1}) = N_1$ and $x_n = c_n$. We now wish to find $x_1, \cdots, x_{n-1}$ to satisfy $f_{n-1} = N_1$, or

$$f(f_{n-2}, x_{n-1}) = N_1;$$

and we know that this has a unique solution, say $f_{n-2} = N_2$ and $x_{n-1} = c_{n-1}$. It is

clear that this process continues until we have values for $x_n, x_{n-1}, \cdots, x_3$, and want values of x_1 and x_2 to satisfy

$$f_2(x_1, x_2) = N_{n-2},$$

which is the equation (1) itself. Hence equation (5) gives the required generalization, F being the polynomial defined by (3) and (4).

Putting $n=3$, we find that the equation

$$(x+y)^4 - 6x^3 - 14x^2y - 10xy^2 - 2y^3 + 4z(x+y)^3 + 7x^2 - y^2 + 2xy + 4z^2 - 12xz - 4yz + 6x + 2y + 4z = 8N$$

has, for any positive integer N, one and only one solution for x, y, z in positive integers.

Also solved by G. B. Huff and the proposer.

Barycentric Coördinates

E 495 [1941, 635]. *Proposed by Daniel Arany, Budapest, Hungary*

If x, y, z are the barycentric coördinates of a point Q with respect to a triangle ABC, show that, for any point P in the same plane,

$$xAP^2 + yBP^2 + zCP^2 = xAQ^2 + yBQ^2 + zCQ^2 + (x+y+z)PQ^2.$$

Solution by Howard Eves, Allen Academy, Bryan, Texas

Let A', B', C' be the projections of A, B, C on the line QP, and let k be the line through Q perpendicular to QP. Then we have

$$AP^2 = AQ^2 + PQ^2 - 2QP \cdot QA',$$
$$BP^2 = BQ^2 + PQ^2 - 2QP \cdot QB',$$
$$CP^2 = CQ^2 + PQ^2 - 2QP \cdot QC'.$$

Multiplying these equations by x, y, z, respectively, and adding, we have

$$\sum (xAP^2) = \sum (xAQ^2) + \sum xPQ^2 - 2QP \cdot \sum (xQA')$$
$$= \sum (xAQ^2) + \sum xPQ^2 - 2QP \cdot \sum (x\alpha),$$

where α is the distance from k to A, *etc.* But, from a fundamental property of barycentric coördinates, we know that $\sum(x\alpha) = 0$. Hence the theorem.

(This is easily generalized to any system of points $A, B, C, \cdots$ and their mean center Q for multiples $x, y, z, \cdots$. See, *e.g.*, art. 55 in McClelland's *Geometry of the Circle*.)

Also solved by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4040. *Proposed by N. A. Court, University of Oklahoma*

Given n points $A_1, A_2, \cdots, A_n$ in space, let G denote their centroid, k^2 the sum of the squares of the $n(n-1)/2$ segments determined by the given points, and P_{ij} the power of a given point P with respect to the sphere having for diameter the segment A_iA_j; show that

$$\sum P_{ij} = n(n-1)PG^2/2 - k^2/2n.$$

4041. *Proposed by Cezar Coşniţă, Focşani, Roumania*

Integrate the following differential equations

$$y' = \frac{x^2 - y^2 + 1}{x^2 - y^2 - 1}, \qquad y' = -\frac{y(2x + y - 1)}{x(x + 2y - 1)}.$$

Determine the integral curves passing through the origin.

4042. *Proposed by Henry Scheffé, Princeton University*

Prove that if A is a fixed positive definite hermitian matrix and X is a variable non-negative hermitian matrix (rank = index), then the minimum value of the determinant $|A+X|$ is $|A|$ and is attained if and only if $X=0$.

4043. *Proposed by H. F. Sandham, Trinity College, Dublin*

Prove that the angle in which the major auxiliary circle of a conic inscribed in a triangle cuts the nine point circle, is equal to the angle which the foci of the conic subtend at the inverse of one of them in the circumcircle. Complete this result and deduce that the minor auxiliary circle of the conic which has the Brocard points as foci, touches the nine point circle, and the major auxiliary circle cuts the latter in an angle which is the complement of three times the Brocard angle of the triangle.

4044. *Proposed by V. Thébault, San Sebastián, Spain*

Determine the straight lines such that the circumsphere of the pedal tetrahedron of each of its points with respect to any given tetrahedron $ABCD$ passes through a fixed point P.

Examine the case for which $ABCD$ is orthocentric and P is the foot of one of its altitudes.

SOLUTIONS

Perspective Triangles (Tetrahedrons)

3988 [1941, 213]. *Proposed by N. A. Court, University of Oklahoma*

The symmetrics of a given straight line (plane) with respect to the sides (faces) of a given triangle (tetrahedron) form a second triangle (tetrahedron) perspective to the first, and the center of perspectivity is equidistant from the sides (faces) of the second triangle (tetrahedron).

Solution by H. W. Eves, Allen Academy, Bryan, Texas

We shall prove the theorem for the plane case; a parallel proof can be given for the solid case. Let ABC be the given triangle with sides a, b, c and let l be the given line. Let the reflections of l in a, b, c be a', b', c', giving the second triangle $A'B'C'$. Now triangles ABC and $A'B'C'$ are centrally perspective because they are axially perspective (on l). All that remains to be proved, then, is that AA', BB', CC' are bisectors of interior or exterior angles of triangle $A'B'C'$. To this end consider triangle $b'c'l$. In this triangle b and c are interior or exterior angle bisectors because of the reflection property. It follows that AA' must be a third angle bisector, bisecting the interior or exterior angle at A'. Similar remarks now hold for BB' and CC' and the theorem is proved.

Solved also by the proposer in a similar manner giving the detailed proof for the tetrahedron and remarking that the proof for the triangle is analogous.

Cones and Spheres

3989 [1941, 213]. *Proposed by N. A. Court, University of Oklahoma*

Three given spheres with non-collinear centers are touched by a (fourth) sphere in the points P, Q, R, and (p), (q), (r) are great circles, in parallel planes, on the three given spheres. Show that the three cones $P(p)$, $Q(q)$, $R(r)$ have a circle in common.

Solution by H. W. Eves, Allen Academy, Bryan, Texas

Given two spheres S_M and T tangent at a point M. Let (m) be a great circle of S_M. Then it is a simple matter to show that the cone $M(m)$ cuts T in a great circle (t) whose plane is parallel to the plane of (m). The theorem now readily follows. For let S_P, S_Q, S_R be the three given spheres and T the fourth tangent sphere. Then, since the planes of (p), (q), (r) are parallel, the three cones $P(p)$, $Q(q)$, $R(r)$ all cut T in the great circle (t) whose plane is parallel to that of (p), (q), (r). This proves the theorem. Note that the condition of non-collinearity of the centers of S_P, S_Q, S_R is not necessary.

Solved also by G. A. Yanosik and the proposer.

Editorial Note. The solution by the proposer is similar to the above and gives essentially the following argument:

Since M is the point of contact of the two spheres S_M and T, it is a homothetic center for the two spheres; and in the homothetic relation thus defined the two centers are corresponding points. To the plane of the great circle (m) there

corresponds a diametral plane of T; and hence the cone $M(m)$ cuts the sphere T in points of this diametral plane of T.

The solution by Yanosik is analytic and much more complicated than the above simple synthetic proof.

Envelope of Spheres

3992 [1941, 214]. *Proposed by V. Thébault, San Sebastián, Spain*

Show that the envelope of a variable sphere (S) which has its center on a quadric surface of revolution (Q) and which is orthogonal to a sphere (Σ) tangent to (Q) along a circle (C) is composed of two spheres passing through (C).

Solution by G. A. Yanosik, New York University

Since the entire configuration here involved is one of revolution about a given axis, the problem may be solved by working in any plane section taken through the axis of revolution.

Let the Y-axis be the axis of revolution, and let the plane $z=0$ be the plane of section.

Take the section of (Q) as

$$x^2 = ay^2 + by + c.$$

The diameter of circle (C) will join the points $(\pm\sqrt{c}, 0)$.
The section of (Σ) will now be

$$x^2 + y^2 - by - c = 0.$$

Now the section of the variable sphere (S) will be given by

$$(x - h)^2 + (y - k)^2 = R^2,$$

subject to

$$h^2 = ak^2 + bk + c,$$

and to

$$R^2 + c = h^2 + k^2 - bk.$$

These three relations lead to

$$[(b - 2y)^2 - 4ax^2]k^2 + 2[(b - 2y)(x^2 + y^2 + c) - 2bx^2]k + [(x^2 + y^2 + c)^2 - 4cx^2] = 0.$$

The parameter being k, the envelope will now be found to be

$$a[x^2 + y^2] + [b \pm \sqrt{(1 + a)(b^2 - 4ac)}]y - ac = 0,$$

which equations represent two circles passing through the points $(\pm\sqrt{c}, 0)$. This establishes the truth of the final statement in the problem.

Editorial Note. The proposer stated that the theorem of the problem results immediately from the similar theorem for the plane; and he also remarked that,

when the quadric surface is a paraboloid of revolution, the envelope of (S) is the plane of the circumference of (C). No proofs were given.

Some properties of the envelope of the plane figure are easily found when the conic is replaced by any curve (Q) which is tangent to a fixed circle (C) at the point T. A circle (S) with its center S on (Q) is orthogonal to (C), and we consider the envelope of (S). A given circle (S) and a similar neighboring circle intersect in two points which are inverses with respect to (C). The limiting positions E_1 and E_2 of these two points on (S) are points of the envelope, and it is clear that E_1 and E_2 are inverses with respect to (C) and that the midpoint P of E_1E_2 is on the pedal curve of (Q) with respect to C, the center of (C). If E_1 and E_2 are real and distinct points, the point P must lie outside the circle (C); when S is at T, the three points P, E_1, E_2 must coincide in T. Thus the envelope is invariant under inversion with respect to (C) and it must contain the point T, which may be a singular point for the envelope. If (Q) cuts (C) in a point, the two curves not being tangent at the point, the corresponding point P for the intersection is inside (C), and E_1 and E_2 are imaginary. If a portion of (Q) lies inside (C), then each of the points of this portion yields only imaginary points for the envelope, except for points, such as T, where this portion may be tangent to (C). An example of this is the case where (Q) is an ellipse inside (C) and tangent to (C) at one or two points.

Suppose now that a certain portion of (Q), proceeding in a given sense from T, is such that the tangent at each of its points contains no interior point of (C). Then the corresponding parts of the envelope are real. For in this case P lies outside (C), and the circle (CS) with the diameter CS passes through P and cuts (C) in the real points M_1 and M_2. The points C, M_1, P, M_2 are necessarily in this order, the circle (S) must contain P in its interior, and E_1 and E_2 must be real points. Thus, if (Q) is an ellipse containing the circle (C) in its interior so that the two are tangent at one or two points, then all the points E_1 and E_2 are real.

If proceeding from T, say to the right, a portion of (Q) is separated from (C) by the common tangent at T, for example if (C) lies below the tangent, and the tangent at each point S has an increasing inclination to the common tangent, then for each S on this portion the tangent to (Q) cuts (C) in two real points, the point P is inside (C), and E_1 and E_2 are imaginary. This continues until we reach a point S_1 on (Q) such that the tangent at S_1 is also tangent to (C) at C_1, and then for S_1 the three points P, E_1, E_2 coincide in C_1. For points on this portion of (Q) beyond S_1, to a certain extent the corresponding points P lie outside (C) and inside the corresponding circles (S), and the corresponding points E_1 and E_2 are real. Thus if (C) is tangent to the two branches of a hyperbola, the envelope has no real points except the two points of contact.

If (Q) is a parabola tangent to the circle (C) at two distinct points, the envelope is real and passes through these two points of contact. From the above solution we see that one part of the envelope is the straight line of the common chord; hence the other part is the inverse of this straight line, which is the circle

through C and the two points of contact. The equation of this circle in the notation of the solution is

$$2b(x^2 + y^2) + (4c - b^2)y - 2bc = 0, \qquad a = 0, \qquad bc \neq 0.$$

The equation of the envelope in the solution results after discarding the factor x^2. The equation of the axis on the conic is $x = 0$; and there cannot be more than four points of the envelope on this axis, and these points must be also on the two circles of the envelope. Hence the factor x^2 should be discarded.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to C. O. Oakley, Haverford College, Haverford, Pennsylvania.

The University of Michigan is offering a new course in mathematics under Professor H. C. Carver, who has been serving as a cadet in the Army Air Force since February. The course is designed to speed up the training of navigators for the air force and is being given during the first eight weeks of the summer session beginning June 15. From June 29 to August 21, also at the University of Michigan, Professor Jerzy Neyman of the University of California is offering an advanced course in the theories of testing hypotheses and of estimation, and he also joins with Professor C. C. Craig in conducting the regular seminar in mathematical statistics for advanced students.

The graduate school of New York University has instituted a program beginning June 23, 1942, the main purpose of which is to enable college graduates to obtain their master's degree in scientific fields by February 1943, so that they may become available as scientific workers or specialized personnel for the war effort. During the summer two half-courses in advanced calculus including vector analysis are given by Professor Friedrichs and Dr. Robbins, and a full course on topics in mathematical physics and applied mathematics by Professor Courant. In addition there is an advanced seminar in research problems.

Fordham University is offering during the summer session an evening course including topics in algebra, solid geometry and plane and spherical trigonometry to prepare prospective candidates for training as officers in the armed forces.

Assistant Professor C. B. Allendoerfer of Haverford College has been promoted to an associate professorship.

At Massachusetts Institute of Technology Assistant Professor P. D. Crout has been promoted to an associate professorship and Dr. Eric Reissner to an assistant professorship.

Dr. J. A. Daum of the A. and M. College of Texas is now a Second Lieutenant, U. S. Army.

Dr. W. B. Fite, Davies professor of mathematics at Columbia University, retired in June.

Assistant Professor J. S. Frame of Brown University has been appointed associate professor and head of the department at Allegheny College.

Dr. J. L. Gibson, professor of mathematics at the University of Utah since 1904 and dean of the School of Arts and Sciences since 1915, retired in June 1941. He continues as president of the Utah Conservation and Research Foundation, an office he has held since 1937.

After forty-seven years of service at Williams College, Professor J. G. Hardy retired at the end of this academic year.

Miss Will Lipscombe, assistant professor at the University of Akron, has been promoted to an associate professorship.

Dr. C. T. McCormick of Fort Hays Kansas State College has been made professor and head of the department.

M. L. Manning, research engineer with the Westinghouse Electric and Manufacturing Company, has been appointed an associate professor of electrical engineering at Illinois Institute of Technology.

Professor J. S. Petersen, Jr., is on leave of absence from Brescia College, New Orleans, and is serving in the U. S. Army Signal Corps.

At the College of the City of New York, Professor F. G. Reynolds retired in June, and Associate Professor Maximilian Philip succeeds him as head of the department of mathematics.

Professor P. R. Rider of Washington University has been appointed an exchange professor at the National University of Mexico for fourteen months beginning August 1, 1942. This appointment is under the Convention for the Promotion of Inter-American Cultural Relations, and under the auspices of the Division of Cultural Relations of the U. S. Department of State. He is to lecture on mathematical statistics.

At Louisiana State University, Professor S. T. Sanders retired June 30, 1942, after thirty-five years of service there, and Professor W. V. Parker succeeds him as head of the department of mathematics.

Assistant Professor C. V. L. Smith of Lafayette College is on leave of absence and is a lieutenant (jg) of class D-V(S), U. S. Naval Reserve. In March through June he attended the Naval Training School of Radio Engineering at Bowdoin College.

Dr. Roy MacKay, associate professor at New Mexico State College, died May 12, 1942, at the age of thirty-eight. He had been a member of the Association for eleven years.

W. F. Reynolds, Chief, Section of Triangulation, Division of Geodesy, U. S. Coast and Geodetic Survey, died May 1, 1942, at the age of sixty-one. He had been a member of the Mathematical Association for fifteen years.

Maria M. Roberts, Professor Emeritus of mathematics at Iowa State College, died on April 12, 1942, at the age of seventy-four. She had taught at Iowa State College for fifty years, and was a charter member of the Mathematical Association.

THE NATIONAL MATHEMATICS MAGAZINE

A campaign is being conducted to increase materially the number of subscribers of the *National Mathematics Magazine*, published at Baton Rouge, Louisiana. This publication, under the editorship of Professor S. T. Sanders, has long received financial support from Louisiana State University. Much of this support is presently to be withdrawn. In working out new financial arrangements an enlarged subscription list is essential. Many mathematicians who do not at present receive the journal will agree that it has an important place in American mathematics and will wish to come to its aid by becoming subscribers. The subscription price is $2.00 per year.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fifth Summer Meeting, Poughkeepsie, N. Y., September 7–9, 1942.
Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, State College, Pa., Oct. 1942
ILLINOIS
INDIANA, Notre Dame, April 9–10, 1943
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI, fall, 1942
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE, fall, 1942
WISCONSIN, Milwaukee, May 7, 1943

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